Topics in Analysis: Example Sheet 4

LENT 08-09 N. WICKRAMASEKERA

(1) Determine the continued fraction expansions of 71/49 and $\sqrt{3}$. Deduce that $\sqrt{3}$ is irrational.

(2) Show that the simple continued fraction representing an irrational number α is uniquely determined by α.

(3) Let a, b be positive integers. Let

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \dots}}}}.$$

(a) Show that x solves $ax^2 + abx - b = 0$.

(b) Assume $a = b = 1$. Show that, in this case, $x = \frac{-1+\sqrt{5}}{2}$ and that if $\frac{p_n}{q_n}$ is the nth convergent of the continued fraction, then $p_n = F_n$, $q_n = F_{n+1}$, where $F_0, F_1, F_2, \dots$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

(c) Show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}$.

(4) Let p be a positive integer and α, β be real numbers such that $\alpha + \beta = \alpha\beta = -p$. Find the simple continued fraction representations of $|\alpha|$ and $|\beta|$ in terms of p.

(5) Determine the rational number with denominator ≤ 10 that best approximates 71/49.

(6) Let $f : [1, \infty) \to \mathbf{R}$ be a continuous function and suppose that for each $x \in [1, \infty)$, $f(nx) \to 0$ as $n \to \infty$, $n \in \mathbf{N}$. Prove that $\lim_{x\to\infty} f(x) = 0$. Hint: For $\epsilon > 0$, consider the sets $Q_k = \{x \in [1, \infty) : |f(nx)| < \epsilon \quad \forall n \geq k\}$.

(7) Is the set of irrationals, as a subset of $\mathbf{R}$, of second category?

(8) Does there exist a sequence of continuous functions $f_n : \mathbf{R} \to \mathbf{R}$ such that the set $\{f_n(x)\}$ is bounded for each irrational x and unbounded for each rational x?

(9) Let A_j be a sequence of subsets of $[0, 1]$ such that for each $N \geq 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0, 1]$. Prove that the set S of points $x \in [0, 1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$?

(10) Use a version of the Baire category theorem to prove that the (standard) Cantor set is uncountable.

(11) If G is an open dense subset of $\mathbf{R}$, and $\mathbf{Q}$ is the set of rationals, show that $G \setminus \mathbf{Q}$ must be dense in $\mathbf{R}$. If we only assume G is uncountable and dense in $\mathbf{R}$, does it still follow that $G \setminus \mathbf{Q}$ is dense in $\mathbf{R}$?

Some optional problems based on the non-examinable material.

(1) Show that a countable union of subsets of $\mathbf{R}$ of Lebesgue measure zero is a set of Lebesgue measure zero.

(2) Show that the Cantor set is an example of an uncountable set of Lebesgue measure zero.

(3) For each $n = 1, 2, \ldots$, let $f_n : [0, 1] \to \mathbf{R}$ be a bounded function. Suppose that f_n is continuous except on a set of Lebesgue measure zero. If $\{f_n\}$ converges uniformly on $[0, 1]$ to some function f, show that f is continuous except on a set of Lebesgue measure zero.

(4) Let $u \in C^2(\mathbf{R}^2)$ satisfy $\Delta u = 0$ in $\mathbf{R}^2$. If $u \equiv 0$ outside some ball $B_R(0)$, prove that $u \equiv 0$ everywhere.

(5) Suppose Ω is a domain in $\mathbf{R}^2$ and u is a C^2 function on Ω which satisfies the mean value property

$$u(y) = \frac{1}{\pi R^2} \int_{B_R(y)} u$$

for every ball $B_R(y)$ with $\overline{B_R(y)} \subset \Omega$. Prove that u is harmonic in Ω.

(6) (a) If u is harmonic in $\mathbf{R}^2$ with $|Du|$ bounded, prove that u must be an affine function.

(b) Find all $C^2(\mathbf{R}^2)$ functions with $|Du|$ bounded and Δu constant in $\mathbf{R}^2$.