Topics in Analysis: Example Sheet 2

Lent 2008-09 N. Wickramasekera

(1) Does there exist a function $f : [0,1] \to \mathbf{R}$ with a discontinuity which can be approximated uniformly on $[0,1]$ by polynomials?

(2) Let $f : [0,1] \to \mathbf{R}$ be a continuous function which is not a polynomial. If p_n is a sequence of polynomials converging uniformly to f on $[0,1]$, and $d_n =$ degree of p_n, prove that $d_n \to \infty$.

(3) Use the mean value theorem to prove that if P is a real polynomial of degree at most n which vanishes at $(n+1)$ distinct real numbers, then P is identically zero.

(4) Suppose $f : [-1,1] \to \mathbf{R}$ is $(n+1)$-times continuously differentiable on $[-1,1]$ and let $J_n = \{x_j \in [-1,1] : j = 0,1,2,\ldots,n\}$ be a set of $(n+1)$ distinct points. Let P_{J_n} be the interpolating polynomial of degree $\leq n$ determined by the requirement $P_{J_n}(x_j) = f(x_j)$ for each $j = 0,1,2,\ldots,n$. Let $\beta_{J_n}(x) = (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)$. Prove that for each $x \in [-1,1]$, there exists $\zeta \in (-1,1)$ such that

$$f(x) - P_{J_n}(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}\,\beta_{J_n}(x).$$

(Hint: If $x = x_j$ this holds trivially. If not, consider $g(y) = f(y) - P_{J_n}(y) - \lambda\beta_{J_n}(y)$ where λ is chosen so that $g(x) = 0$.)

Deduce that if f is infinitely differentiable in $[-1,1]$ and $\sup_{x\in[-1,1]} |f^{(n)}(x)| \leq M^n$ for some fixed constant M and all $n = 1,2,3,\ldots$, then the interpolating polynomials P_{J_n} (for arbitrary choices of sets of interpolation points $J_n = \{x_0^{(n)},\ldots,x_n^{(n)}\} \subset [-1,1]$) converge uniformly to f on $[-1,1]$ as $n \to \infty$.

(5) Fix $n \geq 1$ and let J be any set of n distinct pints $\{x_1,\ldots,x_n\} \subset [-1,1]$. Let β_J be the polynomial defined by $\beta_J(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$ and set

$$F(x_1,\ldots,x_n) = \sup_{x\in[-1,1]} |\beta_J(x)|.$$

By considering the nth Chebychev polynomial or otherwise, prove that F is minimized when $x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right)$ for $k = 1,2,\ldots,n$.

(6) Recall that the converse of the equal ripple criterion holds. That is to say, if $f \in C([0,1])$ and p is a polynomial which minimizes $\|f-q\|_\infty = \sup_{x\in[0,1]} |f(x)-q(x)|$ among all polynomials q of degree at most n, then there exist $(n+2)$ distinct points $0 \leq x_1 < x_2 < \ldots < x_{n+2} \leq 1$ such that either $f(x_j) - p(x_j) = (-1)^j\|f-p\|_\infty$ for all $j = 1,2,\ldots$ or $f(x_j) - p(x_j) = (-1)^{j+1}\|f-p\|_\infty$ for all $j = 1,2,\ldots$. Assuming this, prove that for any given $f \in C([0,1])$ and each positive integer n, the minimizer of $\|f-q\|_\infty$ among all polynomials q of degree at most n is unique.

(7) Determine all linear operators $L : C([0,1]) \to C([0,1])$ which satisfy (i) $Lf \geq 0$ for all non-negative $f \in C([0,1])$ and (ii) $Lf = f$ for the three functions $f(x) = 1, x, x^2$.

(8) Let $B_n : C([0,1]) \to C([0,1])$ be the Bernstein operator defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Show directly that $B_n f \to f$ uniformly on $[0,1]$ for the function $f(x) = x^3$.

(9) Give a proof of the Weierstrass approximation theorem by completing the following argument. Let $0 < a < b < 1$, and $f : [a,b] \to \mathbf{R}$ be the continuous function we wish to approximate uniformly on $[a,b]$ by polynomials. Extend f to all of $\mathbf{R}$ as a continuous function such that the extended function agrees with f on $[a,b]$ and is identical to zero outside $[0,1]$. Again call the extended function f, and let $M = \sup |f|$.

(a) For each $\delta \in (0,1/2)$ and each $n = 1,2,3,\ldots$, set $I_n = \int_0^1 (1-t^2)^n \, dt$ and $I_{n,\delta} = \int_\delta^1 (1-t^2)^n \, dt$. Show that $I_n > (1+n)^{-1}$ and $I_{n,\delta} < (1-\delta^2)^n$. Thus, for any fixed $\delta \in (0,1/2)$, $I_{n,\delta}/I_n \to 0$ as $n \to \infty$.

(b) Choose numbers a_1, b_1 such that $0 < a_1 < a < b < b_1 < 1$, and set, for $x \in \mathbf{R}$ and $n = 1,2,3,\ldots$,

$$\widetilde{p}_n(x) = \int_{a_1}^{b_1} f(y)(1-(y-x)^2)^n \, dy.$$

Given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x+t) - f(x)| < \epsilon$ for $|t| < \delta$ and any x. Why? Use this fact and a change of variables in the integral above to show that for $x \in [a,b]$,

$$\widetilde{p}_n(x) = 2f(x)(I_n - I_{n,\delta}) + R_n(x)$$

where $|R_n(x)| \le 2\epsilon I_n + 2M I_{n,\delta}$.

(c) Set $p_n = (2I_n)^{-1}\widetilde{p}_n$. Check that p_n is a polynomial of degree $\le 2n$, and that $\sup_{x\in[a,b]} |p_n(x) - f(x)| < 2\epsilon$ for all sufficiently large n.

(10) Calculate the first five Chebychev polynomials.

(11) Calculate the first four Legendre polynomials. Do it both using the formula and using orthogonality and check that your answers agree.

(12) Let Ω be a bounded, open subset of $\mathbf{R}^n$ and $g : \Omega \times (a,b) \to \mathbf{R}$ be a function such that

(i) $g(\cdot, t)$ is bounded and continuous in Ω for each $t \in (a,b)$ and

(ii) $\frac{\partial g}{\partial t}(x,t)$, $\frac{\partial^2 g}{\partial t^2}(x,t)$ exist for each $x \in \Omega$ and $t \in (a,b)$, and $|\frac{\partial^2 g}{\partial t^2}(x,t)| \le M$ for some fixed $M \in (0,\infty)$.

Let $F(t) = \int_\Omega g(x,t)\, dx$. Prove that F is differentiable in (a,b) and for each $t \in (a,b)$,

$$F'(t) = \int_\Omega \frac{\partial g}{\partial t}(x,t)\, dx.$$

(13) For $t \geq 0$ and $x \in \mathbf{R}$ define

$$g(x,t) = \begin{cases} x & \text{if } 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \text{if } \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and let $g(x,t) = -g(x,-t)$ if $t < 0$. Prove that g is continuous on $\mathbf{R}^2$ and that $\frac{\partial g}{\partial t}(x,0) = 0$ for all x. Let $F(t) = \int_{-1}^{1} g(x,t)\,dx$. Prove that $F(t) = t$ if $|t| < 1/4$. Thus $F'(0) \neq \int_{-1}^{1} \frac{\partial g}{\partial t}(x,0)\,dx$.

(14) If $f \in C([0,1])$ and $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0, 1, 2\ldots$, prove that f is the zero function. If we assume that $f \in C([0,1])$ and $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 1, 2, \ldots$, does it still follow that f is the zero function? If we only assume that f is bounded and Riemann integrable on $[0,1]$, and that $\int_0^1 f(x)x^n = 0$ for $n = 1, 2, 3, \ldots$, does it still follow that f is the zero function? If not, can a point x where $f(x) \neq 0$ be a point of continuity of f?

(15) Determine all continuous functions $f : \mathbf{R} \to \mathbf{R}$ having the property that $\int_{-\infty}^{\infty} f(x)\varphi''(x)\,dx = 0$ for each smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ vanishing outside some compact interval.