Topics in Analysis: Example Sheet 4

LENT 2007-08 N. WICKRAMASEKERA

(1) Determine the continued fraction expansions of $71/49$ and $\sqrt{3}$. Deduce that $\sqrt{3}$ is irrational.

(2) Let α be a positive number having the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 \cdots}}}$$

where $a_0, a_1, a_2, \ldots$ are integers. Prove that the nth convergent

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_n}}}}$$

of this expansion is closer to α than any other rational number with denominator $\leq q_n$. That is to say,

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \left| \alpha - \frac{p}{q} \right|$$

whenever p and q are integers with $1 \leq q \leq q_n$.

(3) Determine the rational number with denominator ≤ 10 that best approximates $71/49$.

(4) Let a_j, b_j be positive integers for $j = 0, 1, 2, \ldots$, and set

$$\frac{p_n}{q_n} = a_0 + \cfrac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \ldots + \frac{}{a_{n-1} + \frac{b_{n-1}}{a_n}}}}}$$

for $n = 1, 2, 3, \ldots$. Prove that for appropriate choices of integers p_n, q_n we have for $n = 1, 2, 3, \ldots$ that

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

and deduce that

$$p_{n+1} = a_{n+1} p_n + b_n p_{n-1},$$

$$q_{n+1} = a_{n+1} q_n + b_n q_{n-1}.$$

(5) Let $f : [1, \infty) \to \mathbf{R}$ be a continuous function and suppose that for each $x \in [1, \infty)$, $f(nx) \to 0$ as $n \to \infty$, $n \in \mathbf{N}$. Prove that $\lim_{x \to \infty} f(x) = 0$. Hint: For $\epsilon > 0$, consider the sets $Q_k = \{x \in [1, \infty) : |f(nx)| < \epsilon \quad \forall n \geq k\}$.

(6) Is the set of irrationals, as a subset of $\mathbf{R}$, of second category?

(7) Does there exist a sequence of continuous functions $f_n : \mathbf{R} \to \mathbf{R}$ such that the set $\{f_n(x)\}$ is bounded for each irrational x and unbounded for each rational x?

(8) Let A_j be a sequence of subsets of $[0,1]$ such that for each $N \geq 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0,1]$. Prove that the set S of points $x \in [0,1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$?

(9) Let X and Y be topological spaces and let $f : X \to Y$. If $x \in X$, we say that f is continuous at x if for every open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq V$. Prove that f is continuous if and only of f is continuous at every $x \in X$.

(10) (Another problem on the Weierstrass approximation theorem). (a) If $f : (0,1) \to \mathbf{R}$ is continuous, prove that there exists a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of $(0,1)$.

(b) If $f : (0,1) \to \mathbf{R}$ is continuous and bounded, prove that there exists a sequence of polynomials Q_n such that Q_n are uniformly bounded and $Q_n \to f$ uniformly on compact subsets of $(0,1)$.

Some problems on the non-examinable material.

(1) Let $u \in C^2(\mathbf{R}^2)$ satisfy $\Delta u = 0$ in $\mathbf{R}^2$. If $u \equiv 0$ outside some ball $B_R(0)$, prove that $u \equiv 0$ everywhere.

(2) Suppose Ω is a domain in $\mathbf{R}^2$ and u is a C^2 function on Ω which satisfies the mean value property

$$u(y) = \frac{1}{\pi R^2} \int_{B_R(y)} u$$

for every ball $B_R(y)$ with $\overline{B_R(y)} \subset \Omega$. Prove that u is harmonic in Ω.

(3) (a) If u is harmonic in $\mathbf{R}^2$ with $|Du|$ bounded, prove that u must be an affine function.

(b) Find all $C^2(\mathbf{R}^2)$ functions with $|Du|$ bounded and Δu constant in $\mathbf{R}^2$.

(4) We say a C^2 function on a domain $\Omega \subset \mathbf{R}^2$ is subharmonic (resp. superharmonic) in Ω if $\Delta u \geq 0$ (resp. $\Delta u \leq 0$) in Ω. Generalize the strong maximum principle (proved in lecture for harmonic functions) as follows: A subharmonic (resp. superharmonic) function in Ω cannot attain a maximum (resp. minimum) value in Ω unless it is constant. (Hint: start by generalizing the mean value equality to appropriate mean value inequalities.)

(5) Let D be the closed unit disk in $\mathbf{R}^2$, $\Gamma \subset \mathbf{R}^3$ be a Jordan curve (i.e. a homeomorphic image of the unit circle in $\mathbf{R}^2$) and $M = \varphi_\Gamma(D)$ be the area minimizing surface given by the classical solution φ_Γ to the Plateau problem on D. If Γ is contained in a horizontal slab T (i.e. the region between two planes $z = a$ and $z = b$), prove that $M \subset T$.

(6) Let the D, Γ, M be as in problem (5), and B be a closed ball in $\mathbf{R}^3$. Prove that if $\Gamma \subset \partial B$, then $M \subset B$. (Hint: use the maximum principle of problem (4) above.)