Topics in Analysis: Example Sheet 2

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(1) Does there exist a function $f : [0,1] \to \mathbf{R}$ with a discontinuity which can be approximated uniformly on $[0,1]$ by polynomials?

(2) Let $f : [0,1] \to \mathbf{R}$ be a continuous function which is not a polynomial. If p_n is a sequence of polynomials converging uniformly to f on $[0,1]$, and $d_n =$ degree of p_n, prove that $d_n \to \infty$.

(3) Use the mean value theorem to prove that if P is a real polynomial of degree at most n which vanishes at $(n+1)$ distinct real numbers, then P is identically zero.

(4) Suppose $f : [-1,1] \to \mathbf{R}$ is $(n+1)$-times continuously differentiable on $[-1,1]$ and let $J_n = \{x_j \in [-1,1] : j = 0,1,2,\ldots,n\}$ be a set of $(n+1)$ distinct points. Let P_{J_n} be the interpolating polynomial of degree $\leq n$ determined by the requirement $P_{J_n}(x_j) = f(x_j)$ for each $j = 0,1,2,\ldots,n$. Let $\beta_{J_n}(x) = (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)$. Prove that for each $x \in [-1,1]$, there exists $\zeta \in (-1,1)$ such that

$$f(x) - P_{J_n}(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}\,\beta_{J_n}(x).$$

(Hint: If $x = x_j$ this holds trivially. If not, consider $g(y) = f(y) - P_{J_n}(y) - \lambda\beta_{J_n}(y)$ where λ is chosen so that $g(x) = 0$.)

Deduce that if f is infinitely differentiable in $[-1,1]$ and $\sup_{x\in[-1,1]} |f^{(n)}(x)| \leq M^n$ for some fixed constant M and all $n = 1,2,3,\ldots$, then the interpolating polynomials P_{J_n} (for arbitrary choices of sets of interpolation points $J_n = \{x_0^{(n)},\ldots,x_n^{(n)}\} \subset [-1,1]$) converge uniformly to f on $[-1,1]$ as $n \to \infty$.

(5) Fix $n \geq 1$ and let J be any set of n distinct pints $\{x_1,\ldots,x_n\} \subset [-1,1]$. Let β_J be the polynomial defined by $\beta_J(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$ and set

$$F(x_1,\ldots,x_n) = \sup_{x\in[-1,1]} |\beta_J(x)|.$$

By considering the nth Chebychev polynomial or otherwise, prove that F is minimized when $x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right)$ for $k = 1,2,\ldots,n$.

(6) It can be shown that the converse of the equal ripple criterion holds. That is to say, if $f \in C([0,1])$ and p is a polynomial which minimizes $\|f-q\|_\infty = \sup_{x\in[0,1]} |f(x)-q(x)|$ among all polynomials q of degree at most n, then there exist $(n+2)$ distinct points $0 \leq x_0 < x_1 < x_2 < \ldots < x_{n+1} \leq 1$ such that either $f(x_j) - p(x_j) = (-1)^j\|f-p\|_\infty$ for all $j = 0,1,2,\ldots$ or $f(x_j) - p(x_j) = (-1)^{j+1}\|f-p\|_\infty$ for all $j = 0,1,2,\ldots$. Assuming this, prove that for any given $f \in C([0,1])$ and each positive integer n, the minimizer of $\|f-q\|_\infty$ among all polynomials q of degree at most n is unique.

(7) Determine all linear operators $L : C([0,1]) \to C([0,1])$ which satisfy (i) $Lf \geq 0$ for all non-negative $f \in C([0,1])$ and (ii) $Lf = f$ for the three functions $f(x) = 1, x, x^2$.

(8) Let $B_n : C([0,1]) \to C([0,1])$ be the Bernstein operator defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Show directly that $B_n f \to f$ uniformly on $[0,1]$ for the function $f(x) = x^3$.

(9) If $f \in C([0,1])$ and $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0, 1, 2 \ldots$, prove that f is the zero function. If we only assume that $f \in C([0,1])$ and $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 1, 2, \ldots$, does it still follow that f is the zero function? What if $f \in C([0,1])$ and $\int_0^1 f(x)x^n\,dx = 0$ for all even n? Odd n?

(10) Calculate the first five Chebychev polynomials.

(11) Calculate the first four Legendre polynomials. Do it both using the formula and using orthogonality and check that your answers agree.

(12) The Chebychev polynomials form an orthogonal system with respect to a certain positive weight function w. That is, $\int_{-1}^{1} T_m(x)T_n(x)w(x)\,dx = 0$ if and only if $n \neq m$. Work out what w should be, and prove orthogonality. (Hint: use an appropriate substitution for x.)

(13) Let $E(0) = 0$ and $E(x) = \exp\left(-1/x^2\right)$ for $x \neq 0$. Show that E is smooth everywhere in $\mathbf{R}$, the nth derivative $E^{(n)}(x) = P_n(1/x)E(x)$ for $x \neq 0$, where P_n is a polynomial, and $E^{(n)}(0) = 0$ for all $n = 1, 2, 3, \ldots$.

(14) Let Ω be a bounded, open subset of $\mathbf{R}^n$ and $g : \Omega \times (a,b) \to \mathbf{R}$ be a function such that

- (i) $g(\cdot, t)$ is bounded and continuous in Ω for each $t \in (a,b)$ and
- (ii) For each $x \in \Omega$ and $t \in (a,b)$, $\frac{\partial g}{\partial t}(x,t)$ and $\frac{\partial^2 g}{\partial t^2}(x,t)$ exist and are bounded and continuous as functions of $x \in \Omega$ for each fixed $t \in (a,b)$.

Let $F(t) = \int_\Omega g(x,t)\,dx$. Prove that F is differentiable in (a,b) and for each $t \in (a,b)$,

$$F'(t) = \int_\Omega \frac{\partial g}{\partial t}(x,t)\,dx.$$

(15) For $t \geq 0$ and $x \in \mathbf{R}$ define

$$g(x,t) = \begin{cases} x & \text{if } 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \text{if } \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$

and let $g(x,t) = -g(x,-t)$ if $t < 0$. Prove that g is continuous on $\mathbf{R}^2$ and that $\frac{\partial g}{\partial t}(x,0) = 0$ for all x. Let $F(t) = \int_{-1}^{1} g(x,t)\,dx$. Prove that $F(t) = t$ if $|t| < 1/4$. Thus $F'(0) \neq \int_{-1}^{1} \frac{\partial g}{\partial t}(x,0)\,dx$.

(16) Determine all continuous functions $f : \mathbf{R} \to \mathbf{R}$ having the property that $\int_{-\infty}^{\infty} f(x)\varphi''(x)\,dx = 0$ for each smooth function $\varphi : \mathbf{R} \to \mathbf{R}$ which is zero outside some compact interval.