Riemann Surfaces Lent 2026: Example Sheet 3

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1. (i) Show that every analytic function on $\mathbb{C}/\Lambda \setminus \{0\}$ is a polynomial in $\wp(z)$ and $\wp'(z)$.

   (ii) Show that if $F(X, Y) \in \mathbb{C}[X, Y]$ is a polynomial such that $F(\wp(z), \wp'(z)) = 0$ for all z, then F is a multiple of $Y^2 - 4\prod_{i=1}^{3}(X - e_i)$ (where $e_i = \wp(\omega_i/2)$).

2. Suppose a is a complex number with $|a| > 1$. Show that any analytic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$ must be constant, but that there is a non-constant meromorphic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$.

3. Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp''(z) = 6\wp(z)^2 + A$, for some constant $A \in \mathbb{C}$. Show that there are at least three points and at most five points (modulo Λ) at which $\wp'$ is not locally injective.

4. With notation as in the previous question, and a a complex number with $2a \notin \Lambda$, show that the elliptic function

$$h(z) = (\wp(z - a) - \wp(z + a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

   has no poles on $\mathbb{C} \setminus \Lambda$. By considering the behaviour of h at $z = 0$, deduce that h is constant, and show that this constant is zero.

5. Recall that an open set $U \subseteq \mathbb{C}$ is a *star domain* if there is $z_0 \in U$ such that, for every $z \in U$, the straight-line segment from z_0 to z is contained in U. Prove that every star domain is simply connected.

6. Show that a regular covering map $\pi\colon \widetilde{X} \to X$ of non-empty, path-connected topological spaces is surjective. Use the monodromy theorem to show that if X is simply connected then π is a homeomorphism.

7. Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an analytic map of complex tori and π_j denotes the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is an analytic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 \circ F = f \circ \pi_1$.

   [Hint: Define F as follows. Choose a point μ in $\mathbb{C}$ such that $\pi_2(\mu) = f\pi_1(0)$. For $z \in \mathbb{C}$, join 0 to z by a path $\gamma\colon [0, 1] \to \mathbb{C}$, and observe that the path $f \circ \pi_1 \circ \gamma$ in $\mathbb{C}/\Lambda_2$ has a unique lift to a path $\tilde{\gamma}$ in $\mathbb{C}$ with $\tilde{\gamma}(0) = \mu$. If we define $F(z) = \tilde{\gamma}(1)$, show that $F(z)$ does not depend on the path γ chosen and that F has the required properties.]

8. Let $f\colon R \to S$ be an analytic map of compact Riemann surfaces, and $B \subset S$ the set of branch points. Given a point $P \in S \setminus B$, explain how a closed curve γ in $S \setminus B$ starting and ending at P defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. [Recall from Sheet 2, Question 3 that $R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map.]

   Determine this subgroup when $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $z^3 - 3z + 1$ (see Sheet 2, Question 2).

9. Let f and F be as in Question 7, and suppose that f is a conformal equivalence. Show that $F(z) = \lambda z + \mu$, for some $\lambda \in \mathbb{C}_*$. Hence deduce that two analytic tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = c\Lambda_1$ for some $c \in \mathbb{C}^*$.

10. For $i = 1, 2$ let $\tau_i \in \mathbb{C}$ with $\mathrm{Im}(\tau_i) > 0$, and $\Lambda_i = \mathbb{Z} + \mathbb{Z}\tau_i \subset \mathbb{C}$. Show that complex tori $\mathbb{C}/\Lambda_i$ are conformally equivalent if and only if

$$\tau_2 = \pm \frac{a\tau_1 + b}{c\tau_1 + d}$$

for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

11. Let $\mathrm{li}_2 \colon D(0,1) \to \mathbb{C}$ be the analytic function given by the series $\mathrm{li}_2(z) = \sum_{n\geq 1} z^n/n^2$, so that $z\, d\mathrm{li}_2(z)/dz = -\log(1-z)$. We call li_2 the *dilogarithm* function. Let $\mathcal{F}$ be the complete analytic function determined by li_2, and $\pi \colon R = R_{\mathcal{F}} \to \mathbb{C}$ its Riemann surface as a covering space of $\mathbb{C}$.

   (i) Show that for any function element $(f, D) \in \mathcal{F}$, $(zf')' = 1/(1-z)$. Deduce that $\pi(R) \subset \mathbb{C} \smallsetminus \{1\}$.

   (ii) Show that $\pi(R) = \mathbb{C} \smallsetminus \{1\}$ but that π is not a regular covering.

12. Show that $\mathbb{C} \smallsetminus \{P, Q\}$, where $P \neq Q$, is not conformally equivalent to $\mathbb{C}$ or $\mathbb{C}_*$, and deduce from the uniformization theorem that it is uniformized by the open unit disc D. Show that the same is true for any domain in $\mathbb{C}$ whose complement has more than one point.

13. Let R be a compact Riemann surface of genus g and let $p_1, \ldots, p_n$ be distinct points of R with $n \geq 1$. Show that $R \smallsetminus \{p_1, \ldots, p_n\}$ is uniformized by the open unit disc D if and only if $2g - 2 + n > 0$, and by $\mathbb{C}$ if and only if $2g - 2 + n = 0$ or -1.