Riemann Surfaces Lent 2026: Example Sheet 2

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1. Find an explicit analytic map of Riemann surfaces $D = D(0,1) \to D \setminus \{0\}$ which is a regular covering.

2. Consider the analytic map $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the ramification points of f and the corresponding ramification indices. What are the branch points?

3. Suppose that $f\colon R \to S$ is an analytic map of compact Riemann surfaces, and let $B \subset S$ denote the set of branch points. Show that the map $f\colon R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map. [Hint: Mimic the proof of the valency theorem.]

4. Let $f(z) = p(z)/q(z)$ be a rational function on $\mathbb{C}$, where p, q are coprime polynomials. Show that f defines an analytic map $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, whose degree d is the maximum of the degrees of p and q. If f' denotes the derivative of the function f, show that it defines an analytic map $f'\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, whose degree satisfies $d-1 \le \deg f' \le 2d$. [Hint: Consider the principal parts of f at its poles.] Give examples to demonstrate that the bounds can be achieved.

5. If $f\colon R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \ge g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

6. Let $\pi\colon R \to \mathbb{C} \setminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe R explicitly by a gluing construction. Assuming the fact that R may be compactified to a compact Riemann surface $\overline{R}$ and π extended to an analytic map $\bar{\pi}\colon \overline{R} \to \mathbb{C}_\infty$, find the genus of $\bar{R}$.

7. Suppose $\Omega \subseteq \mathbb{C}$ is a discrete additive subgroup. [Here, *discrete* means that the subspace topology on Ω is discrete.] Show that one of the following holds:

   (i) $\Omega = \{0\}$, or

   (ii) $\Omega = \mathbb{Z}\omega$ for some $\omega \neq 0$, or

   (iii) $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \neq 0$ and $\omega_2/\omega_1 \notin \mathbb{R}$.

8. Let f be a simply periodic analytic function on $\mathbb{C}$ with periods $\mathbb{Z}$. Suppose furthermore that $f(x+iy)$ converges uniformly in x to (possibly infinite) limits as $y \to \pm\infty$. Show that $f(z) = \sum_{k=-n}^{n} a_k e^{2\pi i k z}$, i.e. $f(z)$ has a *finite* Fourier expansion.

9. Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$, and let $P \subset \mathbb{C}$ be a fundamental parallelogram. Using the argument principle and, if necessary, slightly perturbing P, show that the number of zeros of f in P is the same as the number of poles, both counted with multiplicities.

   [This is also a consequence of the valency theorem, but the point of this question is that this more direct argument via contour integration also works.]

10. With the notation of the previous question, let the degree of f be n, let $a_1, \ldots, a_n$ denote the zeros of f in a fundamental parallelogram P, and let $b_1, \ldots, b_n$ denote the poles (both with possible repeats). By considering the integral

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)} dz$$

and, if required, also slightly perturbing P, show that

$$\sum_{j=1}^{n} a_j \; - \; \sum_{j=1}^{n} b_j \; \in \Lambda \, .$$

11. Let f, g be non-constant meromorphic functions on a compact Riemann surface R. Show that there is a non-zero polynomial $P(w_1, w_2)$ such that $P(f, g) = 0$.

[Hint: Suppose f, g have valencies m, n respectively, and put $d = m + n$. Show that it is possible to choose complex numbers a_{ij}, not all zero, such that the function

$$\sum_{j=0}^{d} \sum_{k=0}^{d} \; a_{jk} f(z)^j g(z)^k$$

has at least $(d^2 + 2d)$ distinct zeros in R. Show that it cannot have more than d^2 poles, and deduce that it must be identically zero on R.]