# Riemann Surfaces

## Example Sheet 2

### Lent 2025

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1. Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic map, thus is of the form $p(z)/q(z)$ where $p$, $q$ are coprime polynomials.

   (i) Show how to find the degree (valence) $\deg(f)$ of $f$.

   (ii) If $f'$ denotes the derivative of the function $f$, show that it defines an analytic map $f' : \mathbb{C}_\infty \to \mathbb{C}_\infty$ whose degree satisfies $\deg(f) - 1 \leq \deg(f') \leq 2\deg(f)$. [Hint: Consider the principal parts of $f$ at its poles.] For any value of $\deg(f) \geq 1$, give examples to demonstrate that both of these bounds can be achieved.

2. Let $\pi : X \to Y$ be a local homeomorphism between topological spaces and suppose that $X$ is connected and Hausdorff. If $f : X \to X$ is a continuous map such that $\pi \circ f = \pi$, show that $f$ has no fixed points unless it is the identity.

3. (i) Suppose that $f : X \to S$ is a local homeomorphism with $X$ a connected Hausdorff topological space and $S$ a Riemann surface. Show that $X$ can be given the structure of a Riemann surface, under which $f$ becomes analytic.

   [*You'll need some charts.*]

   (ii) (*For those that took IB Geometry*) Now suppose that $g : R \to Y$ is a surjective local homeomorphism where $Y$ is a Hausdorff topological space and $R$ is a Riemann surface. Show that $Y$ can be given the structure of a topological surface. Can $Y$ always be made into a Riemann surface? What if $g$ is a covering map?

4. Find an explicit covering map of Riemann surfaces $D \to D'$, where $D$ denotes any open disc of $\mathbb{C}$ and and $D'$ denotes the same disc minus its centre.

5. Consider the analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the branch points $B$ of $f$ and the corresponding multiplicities. What are the critical values? Is $f$ a covering map? If not, how might we remove points from the domain to turn it into one?

6. Suppose that $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces and let $B \subset R$ denote the set of branch points. Given a point $P \in S \setminus f(B)$, explain how a closed curve $\gamma$ in $S \setminus f(B)$ starting and ending at $P$

defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. What group is obtained in the previous question?

7. If $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \geq g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

8. Recall that an open set $U \subseteq \mathbb{C}$ is a *star domain* if there is $z_0 \in U$ such that, for every $z \in U$, the straight-line segment from $z_0$ to $z$ is contained in $U$. Prove that every star domain is simply connected.

9. Suppose that $\pi : \widetilde{X} \to X$ is a covering map of path-connected topological spaces. Use the monodromy theorem to show that if $X$ is simply connected then $\pi$ is a homeomorphism.

10. Show that the component of the space of germs over $\mathbb{C} \setminus \{0\}$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbb{C}$. Show that the component of the space of germs over $\mathbb{C} \setminus \{-1, 1\}$ corresponding to the complete analytic function $(z^2 - 1)^{1/2}$ is analytically isomorphic to the Riemann surface of this function which is obtained by gluing (as was exhibited informally in Section 1).

11. Let $R$ denote the Riemann surface associated with the complete analytic function $\sqrt{1 - \sqrt{z}}$ over $\mathbb{C} \setminus \{0\}$. Show that the projection covering map to $\mathbb{C} \setminus \{0\}$ is surjective. Find analytic continuations along homotopic curves in $\mathbb{C}\setminus$, say from 1/2 to 3/2, which have the same initial germ at 1/2 but different final germs at 3/2. Why is this consistent with the classical monodromy theorem?

12. Let $\pi : R \to \mathbb{C} \setminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe $R$ explicitly by a gluing construction. Assuming the fact that $R$ may be compactified to a compact Riemann surface $\overline{R}$ by adding finitely many points and that $\pi$ may be extended to an analytic map $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$, find the genus of $\overline{R}$.