Riemann Surfaces

Example Sheet 1

Lent 2025

Please send any comments or corrections to me at `jb128@cam.ac.uk`.

1. (Topology warm up)

   Recall that for *open* subsets of $\mathbb{R}^n$, being connected and being path connected are equivalent. Sometimes it is useful to have injective paths, so given any topological space X and distinct points $x, y \in X$ we define an *arc* $\alpha : [0,1] \to X$ from x to y to be a continuous *injective* map with $\alpha(0) = x$ and $\alpha(1) = y$. We say that X is *arc connected* if for any distinct $x, y \in X$ there exists an arc from x to y.

   (i) If we have an equivalence relation on a connected space X such that all equivalence classes are open then show there is only one equivalence class.

   For the rest of the question, X is Hausdorff.

   (ii) Suppose we have three distinct points $x, y, z \in X$ where there is an arc from x to y and an arc from y to z. Show that there exists an arc from x to z.

   (iii) Show that the relation $x \sim y$ given by ($x = y$ or there exists an arc from x to y) is an equivalence relation on X.

   (iv) Conclude that if X is any open subset of $\mathbb{R}^n$ then X being connected, path connected or arc connected are all equivalent.

2. (i) Show that the power series $f(z) = \Sigma z^{2^n}$ has the unit circle as a natural boundary.

   [*Perhaps you've done something similar on a Complex Analysis question*]

   (ii) The same for $g(z) = \Sigma z^{2^n}/2^n$.

   (iii) And again for $h(z) = \Sigma e^{-2^n} z^{4^n}$, but show further that each derivative $h^{(k)}$ has a continuous extension to the closed unit disc.

   [*Recall uniform convergence, the M-test etc*]

3. Show that the power series $f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n$ defines an analytic function on the unit disc D and give a closed formula for this function. Which points on the unit circle T are regular for f and which are singular? Deduce that the function element (f, D) defines a complete analytic function on the domain $\mathbb{C} \smallsetminus \{1\}$, but does not extend to an analytic function on $\mathbb{C} \smallsetminus \{1\}$.

4. (i) Why is there not a holomorphic function $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $e^{h(z)} = z$?

   (ii) If T is the unit circle, can we have a *continuous* function $f : T \to \mathbb{C}$ with $e^{f(z)} = z$? The same for $r : T \to T$ with $r(z)^2 = z$?

   (iii) The unit circle T is also a multiplicative group, ignoring any topology. Show that we cannot even pick r in (ii) to be a homomorphism.

   [*So be wary of* $1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1} = (\sqrt{-1})^2 = -1$.]

5. Show directly that if (g, V) and (h, V) are both analytic continuations of (f, U) along the same path γ then $g \equiv h$ on V, so that $g(\gamma(1)) = h(\gamma(1))$.

6. (i) Suppose the topological space X satisfies all the conditions for a Riemann surface other than being connected. Show that each connected component of X is a Riemann surface.

   (ii) For an open connected subset $U \subset \mathbb{R}^2$, suppose that C is closed in U and consists of isolated points. Show that $U \setminus C$ is also open and connected. (You could do $U \subset \mathbb{R}^n$ for general n but does it work for all n?)

   (iii) The same as (ii) but now let U be a Riemann surface and show $U \setminus C$ is a Riemann surface.

7. By considering the singularity at ∞ or otherwise, show that any injective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ has the form $f(z) = az + b$, for some $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$. Characterise the injective analytic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

8. Let $D \subset \mathbb{C}$ be an open disc and u a harmonic function on D. Define a complex valued function g on D by $g = u_x - iu_y$; show that g is analytic. If z_0 denotes the centre of the disc, define a function f on D by

$$f(z) = u(z_0) + \int_{z_0}^{z} g(w)dw\,,$$

   the integral being taken over the straight line segment from z_0 to z. Show that f is analytic with $f' = g$, and that $u = \mathrm{Re} f$.

9. Suppose u, v are harmonic functions on a Riemann surface R and $E = \{z \in R : u(z) = v(z)\}$. Show that either $E = R$, or E has empty interior. Give an example to show that E does not in general consist of isolated points.

10. Let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ both be sets of four distinct points in $\mathbb{C}_\infty$. Show that any analytic isomorphism

$$f : \mathbb{C}_\infty \setminus \{a_1, a_2, a_3, a_4\} \to \mathbb{C}_\infty \setminus \{b_1, b_2, b_3, b_4\}$$

   extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$. Using your answer to Question 6, find a necessary and sufficient condition for $\mathbb{C} \setminus \{0, 1, a\}$ to be conformally equivalent to $\mathbb{C} \setminus \{0, 1, b\}$, where $a, b \in \mathbb{C} \setminus \{0, 1\}$.

   [*You might want to consider the cross ratio*]