Riemann Surfaces
Example Sheet 3

Michaelmas 2023

Please send any comments or corrections to me at `jb128@cam.ac.uk`.

1. Let γ be any path in S^2 from x_0 to x_1 (maybe $x_0 = x_1$). Assume that γ is not constant and take any y in the image of γ with $y \neq x_0, x_1$. Suppose that D is a small open disc around y such that the closed disc $\overline{D}$ in S^2 misses x_0 and x_1.

   (i) Show that $\gamma^{-1}(D)$ is a collection of (finite or) countably many disjoint subintervals $(a_1, b_1), (a_2, b_2), \ldots$ of $(0, 1)$.

   (ii) Show that $\gamma^{-1}(\{y\})$ lies in only finitely many of these intervals, say $(a_1, b_1), \ldots, (a_k, b_k)$ wlog.

   (iii) For $1 \leq i \leq k$, on restricting γ to the path $\gamma_i : [a_i, b_i] \to S^2$, show that there is a homotopy of γ_i (fixing endpoints) to a path δ_i which misses y.

   (iv) Conclude that there is a homotopy of γ (fixing endpoints) to a path δ that misses y and therefore S^2 is simply connected.

2. Let f be a simply periodic analytic function on $\mathbb{C}$ with periods $\mathbb{Z}$. Suppose furthermore that $f(x + iy)$ converges uniformly in x to (possibly infinite) limits as $y \to \pm\infty$. Show that $f(z) = \sum_{k=-n}^{n} a_k e^{2\pi i k z}$, i.e. $f(z)$ has a *finite* Fourier expansion.

3. Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an analytic map of complex tori and π_j denotes the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 \circ F = f \circ \pi_1$.

   [Hint: Define F as follows. Choose a point μ in $\mathbb{C}$ such that $\pi_2(\mu) = f\pi_1(0)$. For $z \in \mathbb{C}$, join 0 to z by a path $\gamma : [0, 1] \to \mathbb{C}$, and observe that the path $f \circ \pi_1 \circ \gamma$ in $\mathbb{C}/\Lambda_2$ has a unique lift to a path $\tilde{\gamma}$ in $\mathbb{C}$ with $\tilde{\gamma}(0) = \mu$. If we define $F(z) = \tilde{\gamma}(1)$, show that $F(z)$ does not depend on the path γ chosen and that F has the required properties.]

4. Let f and F be as in Question 3, and suppose that f is a conformal equivalence. Show that $F(z) = \lambda z + \mu$, for some $\lambda \in \mathbb{C}_*$. Hence deduce that two analytic tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda\Lambda_1$ for some $\lambda \in \mathbb{C}_*$.

5. Show that two complex tori, $\mathbb{C}/\langle 1, \tau_1 \rangle$ and $\mathbb{C}/\langle 1, \tau_2 \rangle$, are conformally equivalent if and only if
$$\tau_2 = \pm \frac{a\tau_1 + b}{c\tau_1 + d}$$
for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

6. Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$, and let $P \subset \mathbb{C}$ be a fundamental parallelogram. Using the argument principle and, if necessary, slightly perturbing P, show that the number of zeros of f in P is the same as the number of poles, both counted with multiplicities.

   [This is also a consequence of the valency theorem, but the point of this question is that this more direct argument via contour integration also works.]

7. Suppose a is a complex number with $|a| > 1$. Show that any analytic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$ must be constant, but that there is a non-constant meromorphic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$.

8. Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp$ satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$, for some constant $A \in \mathbb{C}$. Show that there are at least three points and at most five points (modulo Λ) at which $\wp'$ is not locally injective.

9. With notation as in the previous question, and a a complex number with $2a \notin \Lambda$, show that the elliptic function
$$h(z) = (\wp(z-a) - \wp(z+a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$
has no poles on $\mathbb{C} \setminus \Lambda$. By considering the behaviour of h at $z = 0$, deduce that h is constant, and show that this constant is zero.

10. Show that $\mathbb{C} \setminus \{P, Q\}$, where $P \neq Q$, is not conformally equivalent to $\mathbb{C}$ or $\mathbb{C}_*$, and deduce that it is uniformized by the open unit disc $\mathbb{D}$. Show that the same is true for any domain in $\mathbb{C}$ whose complement has more than one point.

11. Let R be a compact Riemann surface of genus g and $p_1, \dots, p_n$ be distinct points of R with $n \geq 1$. Show that $R \setminus \{p_1, \dots, p_n\}$ is uniformized by the open unit disc $\mathbb{D}$ if and only if $2g - 2 + n > 0$, and by $\mathbb{C}$ if and only if $2g - 2 + n = 0$ or -1.

12. Let f, g be non-constant meromorphic functions on a compact Riemann surface R. Show that there is a non-zero polynomial $P(w_1, w_2)$ such that $P(f, g) = 0$.

   [Hint: Suppose f, g have valencies m, n respectively, and put $d = m + n$. Show that it is possible to choose complex numbers a_{ij}, not all zero, such that the function
$$\sum_{j=0}^{d} \sum_{k=0}^{d} a_{jk} f(z)^j g(z)^k$$
has at least $(d^2 + 2d)$ distinct zeros in R. Show that it cannot have more than d^2 poles, and deduce that it must be identically zero on R.]