Riemann Surfaces
Example Sheet 2

Michaelmas 2023

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1. Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an analytic map, thus is of the form $p(z)/q(z)$ where p, q are coprime polynomials.

   (i) Show how to find the degree (valence) $\deg(f)$ of f.

   (ii) If f' denotes the derivative of the function f, show that it defines an analytic map $f' : \mathbb{C}_\infty \to \mathbb{C}_\infty$ whose degree satisfies $\deg(f) - 1 \le \deg(f') \le 2\deg(f)$. [Hint: Consider the principal parts of f at its poles.] For any value of $\deg(f) \ge 1$, give examples to demonstrate that both of these bounds can be achieved.

2. Let $\pi : X \to Y$ be a local homeomorphism between connected, Hausdorff topological spaces, and let $f : X \to X$ be a continuous map such that $\pi \circ f = \pi$. Show that f has no fixed points unless it is the identity.

3. (i) Suppose that $f : X \to S$ is a local homeomorphism with X a connected Hausdorff topological space and S a Riemann surface. Show that X can be given the structure of a Riemann surface, under which f becomes analytic.

   (ii) Now suppose that $\pi : R \to Y$ is a covering map where Y is a Hausdorff topological space and R is a Riemann surface. Show that Y can be given the structure of a topological surface. Can Y always be made into a Riemann surface?

   [*You'll need some charts.*]

4. Find an explicit covering map of Riemann surfaces $D \to D'$, where D denotes any open disc of $\mathbb{C}$ and and D' denotes the same disc minus its centre.

5. Consider the analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the branch points B of f and the corresponding multiplicities. What are the critical values? Is f a covering map? If not, how might we remove points from the domain to turn it into one?

6. Suppose that $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces and let $B \subset R$ denote the set of branch points. Given a point $P \in S \smallsetminus f(B)$, explain how a closed curve γ in $S \smallsetminus f(B)$ starting and ending at P defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained

from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. What group is obtained in the previous question?

7. If $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \geq g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

8. Recall that an open set $U \subseteq \mathbb{C}$ is a *star domain* if there is $z_0 \in U$ such that, for every $z \in U$, the straight-line segment from z_0 to z is contained in U. Prove that every star domain is simply connected.

9. Suppose that $\pi : \widetilde{X} \to X$ is a covering map of path-connected topological spaces. Use the monodromy theorem to show that if X is simply connected then π is a homeomorphism.

10. Show that the component of the space of germs over $\mathbb{C} \smallsetminus \{0\}$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbb{C}$. Show that the component of the space of germs over $\mathbb{C} \smallsetminus \{-1, 1\}$ corresponding to the complete analytic function $(z^2 - 1)^{1/2}$ is analytically isomorphic to the Riemann surface of this function which is obtained by gluing (as was exhibited informally in Section 1).

11. Let R denote the Riemann surface associated with the complete analytic function $\sqrt{1 - \sqrt{z}}$ over $\mathbb{C} \smallsetminus \{0\}$. Show that the projection covering map to $\mathbb{C} \smallsetminus \{0\}$ is surjective. Find analytic continuations along homotopic curves in $\mathbb{C}\smallsetminus$, say from $1/2$ to $3/2$, which have the same initial germ at $1/2$ but different final germs at $3/2$. Why is this consistent with the classical monodromy theorem?

12. Let $\pi : R \to \mathbb{C} \smallsetminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe R explicitly by a gluing construction. Assuming the fact that R may be compactified to a compact Riemann surface $\overline{R}$ by adding finitely many points and that π may be extended to an analytic map $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$, find the genus of $\bar{R}$.