Riemann Surfaces

Example Sheet 1

Michaelmas 2023

Please send any comments or corrections to me at `jb128@cam.ac.uk`.

1. If A is an n by n matrix with strictly positive real entries then the Perron - Frobenius theorem tells us that A has a unique eigenvalue which is largest in modulus, this eigenvalue is real and strictly positive and it occurs with multiplicity 1. We look at a variation:

   Let A be any n by n matrix over $\mathbb{C}$ and suppose that its eigenvalues are $\lambda_1, \dots, \lambda_n$ (possibly with repeats). Set $M = \max\{\,|\lambda_1|, \dots, |\lambda_n|\,\}$.

   For $F(z) = \Sigma_{j=1}^{n} \frac{M}{M-\lambda_j z}$, let $f(z) = \Sigma_{k=0} c_k z^k$ be the Taylor series of F around 0.

   (i) Where is F holomorphic and what is the radius of convergence of f? Find a formula for each coefficient c_k in terms of $\lambda_1, \dots, \lambda_n$ and hence in terms of the matrix A and its powers.

   (ii) Now suppose that all entries of A are real and non-negative. By finding an appropriate singular point of f, show that M is an eigenvalue of A.

   (iii) In this case, must M have multiplicity 1? Can A have lots of other eigenvalues with modulus M?

2. (i) Show that the power series $f(z) = \Sigma z^{2^n}$ has the unit circle as a natural boundary.

   [*Perhaps you've done something similar on a Complex Analysis question*]

   (ii) The same for $g(z) = \Sigma z^{2^n}/2^n$.

   (iii) And again for $h(z) = \Sigma e^{-2^n} z^{4^n}$, but show further that each derivative $h^{(k)}$ has a continuous extension to the closed unit disc.

   [*Recall uniform convergence, the M-test etc?*]

3. Show that the power series $f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n$ defines an analytic function on the unit disc D and give a closed formula for this function. Which points on the unit circle T are regular for f and which are singular? Deduce that the function element (f, D) defines a complete analytic function on the domain $\mathbb{C} \setminus \{1\}$, but does not extend to an analytic function on $\mathbb{C} \setminus \{1\}$.

4. (i) Why is there not a holomorphic function $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with $e^{h(z)} = z$?

   (ii) If T is the unit circle, can we have a *continuous* function $f : T \to \mathbb{C}$ with $e^{f(z)} = z$? (You could consider $f(z) + f(\overline{z})$.) The same for $r : T \to T$ with $r(z)^2 = z$?

   (iii) The unit circle T is also a multiplicative group, ignoring any topology. Show that we cannot even pick r in (ii) to be a homomorphism.

   [*So be wary of* $1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1} = (\sqrt{-1})^2 = -1.$]

5. Show directly that if (g, V) and (h, V) are both analytic continuations of (f, U) along the same path γ then $g \equiv h$ on V, so that $g(\gamma(1)) = h(\gamma(1))$.

6. By considering the singularity at ∞ or otherwise, show that any injective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ has the form $f(z) = az + b$, for some $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$. Characterise the injective analytic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

7. Let $D \subset \mathbb{C}$ be an open disc and u a harmonic function on D. Define a complex valued function g on D by $g = u_x - iu_y$; show that g is analytic. If z_0 denotes the centre of the disc, define a function f on D by

$$f(z) = u(z_0) + \int_{z_0}^{z} g(w)dw\,,$$

   the integral being taken over the straight line segment from z_0 to z. Show that f is analytic with $f' = g$, and that $u = \mathrm{Re} f$.

8. Suppose u, v are harmonic functions on a Riemann surface R and $E = \{z \in R : u(z) = v(z)\}$. Show that either $E = R$, or E has empty interior. Give an example to show that E does not in general consist of isolated points.

9. Let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ both be sets of four distinct points in $\mathbb{C}_\infty$. Show that any analytic isomorphism

$$f : \mathbb{C}_\infty \setminus \{a_1, a_2, a_3, a_4\} \to \mathbb{C}_\infty \setminus \{b_1, b_2, b_3, b_4\}$$

   extends to an analytic isomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$. Using your answer to Question 6, find a necessary and sufficient condition for $\mathbb{C} \setminus \{0, 1, a\}$ to be conformally equivalent to $\mathbb{C} \setminus \{0, 1, b\}$, where $a, b \in \mathbb{C} \setminus \{0, 1\}$.

   [*Don't forget the cross ratio from IB - or was it IA?*]

10. Let $f(z)$ be the complex polynomial $z^3 - z$; consider the subspace R of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ given by the equation $w^2 = f(z)$, where (z, w) denote the coordinates on $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that R has the structure of a Riemann surface, on which π is an analytic map. Show that the projection onto the second factor is also an analytic map.

    By deleting three appropriate points from R, show that π yields a covering map from the resulting Riemann surface $R_0 \subset R$ to $\mathbb{C} \setminus \{-1, 0, 1\}$, and that R_0 is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function $(z^3 - z)^{1/2}$ over $\mathbb{C} \setminus \{-1, 0, 1\}$.