Part IID RIEMANN SURFACES (2012–2013)
Example Sheet 1

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Some of the questions below are intended to serve as a refresher on Complex Analysis. Notation: $D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$ is an open disc, and $D^*(a,r) = \{z \in \mathbb{C} \mid 0 < |z-a| < r\}$ is a punctured open disc.

(1) If $f : D^*(a,r) \to \mathbb{C}$ is holomorphic and has a pole of order n at a, show that there exist $\varepsilon > 0$ and $R > 0$ such that for any given w with $|w| > R$, the equation $f(z) = w$ has exactly n distinct solutions z in the punctured disc $D^*(a,\varepsilon)$.

(2) The following is a useful generalization of the argument principle. Let D be an open disc, γ a simple closed curve in D (oriented so that the winding number $n(\gamma, a) = 1$ for a inside γ), f meromorphic, g holomorphic on D, and γ does not pass through any zeros or poles of f. Then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = k_1 g(z_1) + \ldots + k_m g(z_m) - \ell_1 g(w_1) - \ldots - \ell_n g(w_n),$$

where z_j are the zeros of f inside γ, and w_j are the poles of f inside γ, and k_j and ℓ_j are, respectively, their orders. Verify this result. [Hint: no need to factorize g.]

(3) Suppose that f is holomorphic on the open disc $D(a,r)$ and g is a **locally defined inverse** to f at a, i.e. for all w with $|w - f(a)| < \delta$, there is a **unique** $g(w)$ such that $f(g(w)) = w$. Prove that for w near $f(a)$,

$$g(w) = \frac{1}{2\pi i}\int_{\gamma(a,\varepsilon)} z\frac{f'(z)}{f(z)-w}dz,$$

where $\gamma(a,\varepsilon)$ is defined by $\gamma(a,\varepsilon)(t) = a + \varepsilon e^{it}$, $0 \le t \le 2\pi$ (with a suitable $\varepsilon > 0$).
[Hint: apply Q2.]

(4) Show that

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. [Hint: Use the Weierstrass M-test from Analysis II to show that f is locally uniformly convergent.]

(5) (i) Show that a bounded holomorphic function on $D^*(0,1)$ extends holomorphically to all of $D(0,1)$.

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and injective (1:1). Let $F : D^*(0,1) \to \mathbb{C}$ be determined by $F(z) = f(1/z)$. By considering $w \in f(D(0,1))$ and using the Weierstrass–Casorati Theorem, prove that 0 is at worst a pole of F.

(iii) Show that if g is holomorphic on $D^*(0,1)$ and $g(z) = w$ never has more than n solutions z in $D^*(0,1)$ (n is some fixed number) then g has at 0 at worst a pole of order $\leq n$.

(6) Suppose that a holomorphic function f satisfies a polynomial equation

$$f^n(z) + a_{n-1}(z)f^{n-1}(z) + \ldots + a_1(z)f(z) + a_0(z) = 0$$

on a region $U \subset \mathbb{C}$, where the coefficients $a_i(z)$ are holomorphic on $\mathbb{C}$. Show that every analytic continuation of (U, f) also satisfies this equation.

(7) Consider the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \ldots,$$

defined on the open unit disc $D(0,1)$. Prove that the unit circle $\gamma = \{z \in \mathbb{C} \mid |z| = 1\}$ is the natural boundary for the function element $(D(0,1), f)$.

(8) Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with convergent radius $r \in (0, \infty)$. Show that there is at least one singular point on the boundary of $D(a,r)$.

(9) Analytic continuation by reflections. Let f be a function which is holomorphic on the upper half-plane $\mathbb{H}$ and continuous on $\mathbb{H} \cup I$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $f(z) \in \mathbb{R}$ whenever $z \in I$. Prove that $f(z) = \overline{f(\bar{z})}$, for $\mathrm{Im}(z) < 0$, defines an analytic continuation of f to $\mathbb{C} \setminus (\mathbb{R} \setminus I)$.
[Hint: it is convenient to use Morera's theorem from IB Complex Analysis. At some stage, consider a sequence of contours $\gamma_n(t)$, such that the γ_n's converge *uniformly with first derivatives* to a contour $\gamma(t)$ containing a subinterval of $I \subset \mathbb{R}$.]

(10) Show that the unit disc $D(0,1)$ and the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ are conformally equivalent.

(11) Show, by considering the unit disc $D(0,1)$ and the complex plane $\mathbb{C}$, that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic).

(12) Show that no two of the following regions in $\mathbb{C}$ are conformally equivalent

- $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$,
- $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$,
- $\{z \in \mathbb{C} \mid 0 < |z| < \infty\}$

where the complex structures on these sets are those inherited from the usual complex structure on $\mathbb{C}$.

(13) (i) Let X and Y be Riemann surfaces, $f : X \to Y$ a continuous map, and p a point in X. Show, directly from the definition of holomorphic maps, that if f is holomorphic on $X \setminus \{p\}$ then f is in fact holomorphic on all of X.

(ii) Suppose that each of $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of four distinct points in S^2 and $f : S^2 \setminus A \to S^2 \setminus B$ is a biholomorphic map. Show that f extends to a biholomorphic map of S^2 onto itself.

(14) Show that if X and Y are Riemann surfaces such that both are connected, X is compact and Y is **non-compact** then every holomorphic map $f : X \to Y$ is constant.