Part II Riemann Surfaces, Michaelmas 2010 pmhw@dpmms.cam.ac.uk

Example Sheet 2

(1) Let $U \subset \mathbf{C}$ be a star-domain; show that it is simply connected. [Not quite as easy as it looks!]

(2) Let $\pi : \tilde{X} \to X$ be a regular covering map of topological spaces; show that π is surjective. Suppose now that X is simply connected; using the Monodromy theorem, show that π is a homeomorphism.

(3) Suppose that $f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is an analytic map of complex tori, where π_j denotes the projection map $\mathbf{C} \to \mathbf{C}/\Lambda_j$ for $j = 1, 2$. Show that there is an analytic map $F : \mathbf{C} \to \mathbf{C}$ such that $\pi_2 F = f\pi_1$.

[HINT: Define F as follows. Choose a point μ in $\mathbf{C}$ such that $\pi_2(\mu) = f\pi_1(0)$. For $z \in \mathbf{C}$, join 0 to z by a path $\gamma : [0, 1] \to \mathbf{C}$, and observe that the path $f\pi_1\gamma$ in $\mathbf{C}/\Lambda_2$ has a unique lift to a path Γ in $\mathbf{C}$ with $\Gamma(0) = \mu$. If we define $F(z) = \Gamma(1)$, show that $F(z)$ does not depend on the path γ chosen and that F has the required properties.]

(4) If the map f of Question 3 is a conformal equivalence, show that $F(z) = \lambda z + \mu$ for some $\lambda \in \mathbf{C}^*$. Hence deduce that two analytic tori $\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda\Lambda_1$ for some $\lambda \in \mathbf{C}^*$.

(5) Show that complex tori $\mathbf{C}/\langle 1, \tau_1\rangle$ and $\mathbf{C}/\langle 1, \tau_2\rangle$ are analytically isomorphic if and only if $\tau_2 = \pm(a\tau_1 + b)/(c\tau_1 + d)$, for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

(6) Show that the component of the space of germs over $\mathbf{C}^*$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbf{C}$. Show that the component of the space of germs over $\mathbf{C}\backslash\{-1, 0, 1\}$ corresponding to the complete analytic function $(z^3 - z)^{1/2}$ is analytically isomorphic to the Riemann surface we constructed by gluing.

[This is a rather crucial question; do it and you will understand why the abstract construction of Riemann surfaces via the space of germs construction corresponds to the cut and paste constructions. If you are not happy with your answer to this question, make sure that your supervisor goes through it with you.]

(7) Let R denote the Riemann surface associated with the complete analytic function $\sqrt{1 - \sqrt{z}}$ over $\mathbf{C}^*$. Show that the projection covering map to $\mathbf{C}^*$ is surjective. Find analytic continuations along homotopic curves in $\mathbf{C}^*$, say from 1/2 to 3/2, which have the same initial germ at 1/2 but different final germs at 3/2. Why is this consistent with the Classical Monodromy theorem?

(8) Consider the analytic map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the ramification points of f and the corresponding ramification indices. What are the branch points?

(9) Suppose that $f : R \to S$ is an analytic map of compact Riemann surfaces, and let $B \subset S$ denote the set of branch points. Show that the map $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map. [HINT: Similar argument to that used in the Valency theorem.] Given a point $P \in S \setminus B$ and a closed curve γ in $S \setminus B$ with initial and final point P, explain how this defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. What group is obtained in Question 8?

(10) Let $f(z) = p(z)/q(z)$ be a rational function on $\mathbf{C}$, where p, q are coprime polynomials. Show that f defines an analytic map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$, whose degree d is the maximum of the degrees of p and q. If f' denotes the derivative of the function f, show that it defines an analytic map $f' : \mathbf{C}_\infty \to \mathbf{C}_\infty$, whose degree satisfies $d - 1 \leq \deg f' \leq 2d$. [HINT: Consider the principal parts of f at its poles.] Give examples to demonstrate that the bounds can be achieved.

(11) If $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \geq g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

(12) Let $\pi : R \to \mathbf{C} \setminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe R explicitly by a gluing construction. Assuming the fact that R may be compactified to a compact Riemann surface $\bar{R}$ and π extended to an analytic map $\bar{\pi} : \bar{R} \to \mathbf{C}_\infty$, find the genus of $\bar{R}$.