Part II Riemann Surfaces, Michaelmas 2009

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Example Sheet 2

(1) Let $U \subset \mathbf{C}$ be a star-domain; show that it is simply connected.

(2) Let $\pi : \tilde{X} \to X$ be a regular covering map of topological spaces; show that π is surjective. Suppose now that X is simply connected; using the Monodromy theorem, show that π is a homeomorphism.

(3) Suppose that $f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is an analytic map of complex tori, where π_j denotes the projection map $\mathbf{C} \to \mathbf{C}/\Lambda_j$ for $j = 1, 2$. Show that there is an analytic map $F : \mathbf{C} \to \mathbf{C}$ such that $\pi_2 F = f\pi_1$.

[HINT: Define F as follows. Choose a point μ in $\mathbf{C}$ such that $\pi_2(\mu) = f\pi_1(0)$. For $z \in \mathbf{C}$, join 0 to z by a path $\gamma : [0, 1] \to \mathbf{C}$, and observe that the path $f\pi_1\gamma$ in $\mathbf{C}/\Lambda_2$ has a unique lift to a path Γ in $\mathbf{C}$ with $\Gamma(0) = \mu$. If we define $F(z) = \Gamma(1)$, show that $F(z)$ does not depend on the path γ chosen and that F has the required properties.]

(4) If the map f of Question 3 is a conformal equivalence, show that $F(z) = \lambda z + \mu$ for some $\lambda \in \mathbf{C}^*$. Hence deduce that two analytic tori $\mathbf{C}/\Lambda_1$ and $\mathbf{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda\Lambda_1$ for some $\lambda \in \mathbf{C}^*$.

(5) Show that complex tori $\mathbf{C}/\langle 1, \tau_1\rangle$ and $\mathbf{C}/\langle 1, \tau_2\rangle$ are analytically isomorphic if and only if $\tau_2 = \pm(a\tau_1 + b)/(c\tau_1 + d)$, for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

(6) Show that the component of the space of germs over $\mathbf{C}^*$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbf{C}$. Show that the component of the space of germs over $\mathbf{C}\backslash\{-1, 0, 1\}$ corresponding to the complete analytic function $(z^3 - z)^{1/2}$ is analytically isomorphic to the Riemann surface we constructed by gluing.

(7) Let R denote the Riemann surface associated with the complete analytic function $\sqrt{1 - \sqrt{z}}$ over $\mathbf{C}^*$. Show that the projection covering map to $\mathbf{C}^*$ is surjective. Find analytic continuations along homotopic curves in $\mathbf{C}^*$, say from 1/2 to 3/2, which have the same initial germ at 1/2 but different final germs at 3/2. Why is this consistent with the Classical Monodromy theorem?

(8) Consider the analytic map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the ramification points of f and the corresponding ramification indices. What are the branch points?

(9) Suppose that $f : R \to S$ is an analytic map of compact Riemann surfaces, and let $B \subset S$ denote the set of branch points. Show that the map $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map. [Hint: Similar argument to that used in the Valency theorem.] Given a point $P \in S \setminus B$ and a closed curve γ in $S \setminus B$ with initial and final point P, explain how this defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. What group is obtained in Question 8?

(10) Let $f(z) = p(z)/q(z)$ be a rational function on $\mathbf{C}$, where p, q are coprime polynomials. Show that f defines an analytic map $f : \mathbf{C}_\infty \to \mathbf{C}_\infty$, whose degree d is the maximum of the degrees of p and q. If f' denotes the derivative of the function f, show that it defines an analytic map $f' : \mathbf{C}_\infty \to \mathbf{C}_\infty$, whose degree satisfies $d - 1 \leq \deg f' \leq 2d$. Give examples to demonstrate that the bounds can be achieved.

(11) If $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \geq g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

(12) Let $\pi : R \to \mathbf{C} \setminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe R explicitly by a gluing construction. Assuming the fact that R may be compactified to a compact Riemann surface $\bar{R}$ and π extended to an analytic map $\bar{\pi} : \bar{R} \to \mathbf{C}_\infty$, find the genus of $\bar{R}$.