Part IID RIEMANN SURFACES (2008–2009)
Example Sheet 3

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(1) Let $f\colon X \to Y$ be a covering map, in the sense of complex analysis, of compact connected Riemann surfaces. Prove that it is a covering in the sense of topology.

(2) Let $f\colon X \to Y$ be a covering map, in the sense of complex analysis, where Y is a connected Riemann surface and X a connected topological surface. Show that there is a unique complex structure on X with respect to which π is a holomorphic map.

(3) Let X, Y be connected Riemann surfaces. As we have seen in the lectures, a function element (U, f) on X (into Y) determines a complete holomorphic function which corresponds to a connected component $\mathcal{F}$ of the set of all germs $\mathcal{G}$. We also have the map $\pi\colon \mathcal{F} \to X$ given by $\pi([x, g]) = x$ with image $V = \pi(\mathcal{F})$ an open subset of X.

Show that $\mathcal{F}$ satisfies the second countability condition of topology. In particular, $\pi^{-1}\{x\}$ is countable for any $x \in V$. [This completes the proof of $\mathcal{F}$ being a Riemann surface]

(4) Consider the complete holomorphic function on $\mathbb{C}$ determined by $\sqrt{1+\sqrt{z}}$. Show that the corresponding Riemann surface $\mathcal{F}$ contains exactly two germs $[z, f]$ with $z = 1$ and exactly four germs $[z, f]$ for each z such that $0 < |z - 1| < \frac{1}{2}$. [Hint: consider the possible values $f(z)$ for the function elements.]

Let $0 < \epsilon < 1/2$ and consider the associated holomorphic map $\pi\colon \mathcal{F} \to \mathbb{C}$. Verify that the path $\gamma(t) = 1 - \epsilon/2 + \epsilon t,\ \ 0 \le t \le 1$, does not have a lift to $\mathcal{F}$ from $[1 - \epsilon/2, g(1 - h(z))]$, where g, h are holomorphic functions near $1 - \epsilon/2$ and $1 - h(1 - \epsilon/2)$, respectively, satisfying $g(z)^2 = z,\ \ h(z)^2 = z,\ \ h(1) = 1$.

(5) Let X be a connected Riemann surface and $Y = \mathbb{C}$. Show that $\mathcal{G}_x$ the set of all germs at $x \in X$ (into Y) has a 'natural' ring structure. Identify a maximal ideal of this ring.

(6) Prove that an open disc is simply connected. More generally, prove that any convex open subset of $\mathbb{C}$ is simply connected. In particular, then $\mathbb{C}$ is simply connected.

(7) * Try to show that the Riemann sphere is simply connected directly from the definition of simple-connectedness. [Hint: choose a closed path and prove that it is homotopic to a closed path which does not

pass through every point of the Riemann sphere]

(8) Let Λ be a lattice in $\mathbb{C}$. Prove that the quotient map $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is a universal covering map. Conclude that the torus is not simply connected without using the uniformization theorem; but you may use the monodromy theorem.

(9) Let $U \subset \mathbb{C}$ be a region and $a \in U$. Prove that $U \setminus \{a\}$ is not simply connected. [You are allowed to use the uniformization theorem]

(10) Let G be a discrete subgroup of $\mathbb{C}$. Show that G is one of the following:

(i) $\{0\}$,
(ii) $\mathbb{Z}\lambda$, $\lambda \in \mathbb{C}^*$, or
(iii) $\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, $\lambda_i \in \mathbb{C}$, and λ_1, λ_2 are linearly independent over $\mathbb{R}$.

(11) Show, using the uniformization theorem, that any holomorphic map from $\mathbb{C}$ to a compact connected Riemann surface of genus greater than 1 is constant.

(12) Let X be the Riemann sphere. Show that there is no non-trivial subgroup of $\mathrm{Aut}(X)$ acting on X properly discontinuously.

(13) (i) Let $\mathcal{M}$ be the set of complex tori up to conformal equivalence. Show that $\mathcal{M}$ is not countable.
(ii) Let $PSL(2,\mathbb{Z}) = \left\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \det A = 1 \text{ and } a,b,c,d \in \mathbb{Z}\right\} / \pm I$ where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. You can check that $PSL(2,\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$$

Show that there is a bijection between $\mathcal{M}$ and the quotient $\mathbb{H}/PSL(2,\mathbb{Z})$, the set of orbits of $\mathbb{H}$ under the above action.

(14) Show that any holomorphic map f of degree 2 from a complex torus $\mathbb{C}/\Lambda$ to the Riemann sphere is given by a 'Möbius transformation of a shifted $\wp$-function':

$$f(z) = \frac{a\wp(z-z_0)+b}{c\wp(z-z_0)+d},$$

for some $a, b, c, d, z_0 \in \mathbb{C}$.

(15) Show that a compact connected Riemann surface X is a complex torus iff there is a holomorphic map $f\colon X \to Y$ of degree 2 with 4

branch points where Y is the Riemann sphere.

(16) Let $X = \mathbb{C}/\Lambda$ be the complex torus defined by a lattice Λ and write $X_0 = X \setminus \{\Lambda\}$ for the complement of the coset of Λ. Show that

$$\Phi : z + \Lambda \in X_0 \to (\wp(z), \wp'(z)) \in \mathbb{C}^2$$

maps the punctured complex torus X_0 biholomorphically onto a smooth algebraic curve in $\mathbb{C}^2$.

[Hint: the differential equation for $\wp$.]

(17) (i) Let f and g be two elliptic functions (with the same lattice of periods) and N a positive integer. By considering the poles of f and g, estimate from above the dimension of the complex vector space spanned by $f(z)^m g(z)^n$, for $0 \leq m, n \leq N$. Deduce that when N is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \qquad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions f, g on a complex torus $\mathbb{C}/\Lambda$ are **'algebraically related'**: there is a polynomial Q in two variables, so that $Q(f(z), g(z)) = 0$ for all z.

(ii)* Show that in fact (i) holds for meromorphic functions on any compact connected Riemann surface.