Part IID RIEMANN SURFACES (2008–2009)
Example Sheet 2

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(1) Determine the topology on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, that is, determine the open subsets.

(2) Let $p(z, w) \in \mathbb{C}[z, w]$ be a non-constant irreducible polynomial and let X be the algebraic curve $\{(a, b) \in \mathbb{C}^2 \mid p(a, b) = 0\}$ defined by $p(z, w)$. Show that X is not compact.

(3) Let $X = \{(a, b) \in \mathbb{C}^2 \mid b^2 = a^2 - c^2\}$, where c is a fixed non-zero complex number. Show that X is a smooth curve.

By finding the intersection point(s) of X with the complex line $\lambda(a - c) = b$, show that the map $\varphi : \mathbb{C} \setminus \{1, -1\} \to X \setminus \{(c, 0)\}$ given by

$$\varphi(\lambda) = \left(c\frac{\lambda^2 + 1}{\lambda^2 - 1}, \frac{2c\lambda}{\lambda^2 - 1} \right)$$

is biholomorphic. Thus φ can be thought of as a 'parameterization' of an open subset of X.

(4) Let $f \colon X \to Y$ be a continuous map between Riemann surfaces with analytic atlases $\mathcal{A} = \{(U_i, \varphi_i)\}$ and $\mathcal{B} = \{(V_\alpha, \psi_\alpha)\}$ on X and Y respectively. Prove that if f is holomorphic with respect to $\mathcal{A}$ and $\mathcal{B}$, then it is so with respect to any other equivalent atlases.

(5) Let $f \colon X \to Y$ and $g \colon Y \to Z$ be holomorphic maps between Riemann surfaces. Show that the composition map $gf \colon X \to Z$ is a holomorphic map.

(6) Let $f \colon X \to Y$ be a map between Riemann surfaces, and $X = \bigcup_i U_i$ where U_i are open subsets. Show that f is holomorphic if and only if $f|_{U_i} \colon U_i \to Y$ is holomorphic for every i.

(7) Let $f \colon X \to Y$ be a non-constant holomorphic map between connected Riemann surfaces.

(i) Show that $f^{-1}\{y\}$ is a discrete (i.e. with no accumulation points) subset of X for any $y \in Y$. In particular, if X is compact then $f^{-1}\{y\}$ is finite.

(ii) Suppose that $v_f(x) = m$ for some $x \in X$, and $y = f(x)$. Prove that there are open subsets $U \subset X$ and $V \subset Y$ such that $x \in U, y \in V$, and such that $U \cap f^{-1}\{y'\}$ has m elements for any $y \neq y' \in V$.

(iii) Inverse mapping theorem. Suppose that $v_f(x) = 1$ for some $x \in X$. Show that there are open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is biholomorphic and $x \in U$.

(8) Let X be a Riemann surface. A conformal equivalence $f : X \to X$ is called an automorphism of X. Prove that the set of automorphisms of X, denoted by $\mathrm{Aut}(X)$, is a group where the group operation is the composition of maps.

(9) Prove that every $f \in \mathrm{Aut}(\mathbb{C})$ is of the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$ where $a \neq 0$.

(10) Let X be the Riemann sphere. Show that $\mathrm{Aut}(X)$ is isomorphic to $SL(2,\mathbb{C})/\pm I$. Here $SL(2,\mathbb{C})$ is the set of 2×2 matrices over $\mathbb{C}$ with determinant equal to 1, and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(11) (i) Prove Schwartz lemma: if $f : D(0,1) \to D(0,1)$ is holomorphic and $f(0) = 0$, then either $|f(z)| < |z|$, for every $z \in D^*(0,1)$, or $f(z) = e^{i\theta} z$, for some real θ.

(ii) Deduce from Schwartz lemma that any biholomorphic map of $D(0,1)$ onto itself is a Möbius transformation (restricted to $D(0,1)$). You may assume without proof a result (from IB Geometry examples) that a Möbius transformation maps $D(0,1)$ onto itself if and only if it is of the form $z \mapsto \dfrac{az + \bar{c}}{cz + \bar{a}}$, with $|a|^2 - |c|^2 = 1$.
[Hint: reduce the problem to the case when a biholomorphic map of $D(0,1)$ onto itself has a fixed point $z = 0$.]

(iii) Define

$$SU(1,1) = \left\{ A \in GL(2,\mathbb{C}) \mid \det A = 1 \text{ and } A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\overline{A^t} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}.$$

Show that the group $\mathrm{Aut}\, D(0,1)$ is isomorphic to the 'projective special unitary group' $PSU(1,1) = SU(1,1)/\pm I$.

(12) Let $f : X \to Y$ be a non-constant holomorphic map between connected Riemann surfaces. Show that the set of ramification points of f is discrete.

(13) Consider the algebraic curve X in $\mathbb{C}^2$ defined by the vanishing of the polynomial $p(z,w) = w^3 - z(z^2 - 1)$. Show that X is smooth at every point, and find the branch points of $f : X \to \mathbb{C}$ given by the first projection, i.e. $f(z,w) = z$. Find also the ramification points of f and the branching orders.

(14) Let X and Y be compact connected Riemann surfaces and $f : X \to Y$ a non-constant holomorphic map. (Assume that the genus of any

compact connected Riemann surface is a non-negative integer).

(i) Show that the genus of X is greater or equal to the genus of Y.
(ii) If

$$\text{genus}(X) = \text{genus}(Y) > 1$$

show that f is biholomorphic.
(iii) Show that a holomorphic map $f\colon S^2 \to S^2$ of degree $k \geq 2$ must have branch points.

(15) A compact connected Riemann surface X is called **hyperelliptic** if it admits a holomorphic map $f : X \to S^2$ of degree 2. Show that, for any hyperelliptic Riemann surface X, the map $g\colon X \to X$ determined (uniquely) by the properties $f \circ g = f$, and $g(x) \neq x$ if $v_f(x) = 1$, is holomorphic.

(16) Let $f\colon S^2 \to S^2$ be a non-constant holomorphic map, with degree $d \geq 1$. Show that for all but a finite number of points $Q \in S^2$, the equation $f(P) = Q$ has d **distinct** solutions P in S^2. When does $f(P) = Q$ have d distinct solutions for **every** Q?

(17) Analytic continuation by reflections. Let f be a function which is holomorphic on the upper half-plane $\mathbb{H}$ and continuous on $\mathbb{H} \cup I$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $f(z) \in \mathbb{R}$ whenever $z \in I$. Prove that $f(z) = \overline{f(\bar{z})}$, for $\text{Im}(z) < 0$, defines an analytic continuation of f to $\mathbb{C} \setminus (\mathbb{R} \setminus I)$.
[Hint: it is convenient to use Morera's theorem from IB Complex Analysis. At some stage, consider a sequence of contours $\gamma_n(t)$, such that the γ_n's converge *uniformly with first derivatives* to a contour $\gamma(t)$ containing a subinterval of $I \subset \mathbb{R}$.]