Part IID RIEMANN SURFACES (2007–2008): Example Sheet 2

(a.g.kovalev@dpmms.cam.ac.uk)

1. If f is a meromorphic doubly-periodic (i.e. elliptic) function of degree $k > 0$ show that f' is an elliptic function whose degree ℓ satisfies $k + 1 \le \ell \le 2k$. Give examples to show that both bounds are attained.

Recall from example sheet 1: $\psi(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}(n + \frac{1}{2})^2\tau + (n + \frac{1}{2})(z + \frac{1}{2}))$ and satisfies $\psi(z + 1) = -\psi(z)$, $\psi(z + \tau) = -\mathbf{e}(-\frac{\tau}{2} - z)\psi(z)$, where $\mathbf{e}(z) = \exp(2\pi iz)$, $\psi(z) = -\psi(-z)$, and has unique zero 'modulo the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$'.

2. (i) Prove that if $z, w \in \mathbb{C}$, then

$$\wp(z) - \wp(w) = -\psi'(0)^2 \frac{\psi(z - w)\psi(z + w)}{\psi(z)^2\psi(w)^2}.$$

[Hint: Regarding one of w,z as parameter, prove that each side is Λ-periodic in the other variable and has same zeros and poles. Get multiplicative constant by considering Laurent expansion at zero.]

(ii) Deduce that $\wp'(z) = -\psi'(0)^3 \dfrac{\psi(2z)}{\psi(z)^4}$ and recover from this formula the zeros of $\wp'$.

3.* Elliptic functions may be thought of as generalizations of trigonometric functions. To make this more precise, consider $\psi(z, it)$ for $t > 0$. Show that for each fixed z,

$$\exp(\pi t/4)\psi(z, it) \to -2\sin(\pi z), \text{ as } t \to \infty.$$

This suggests the replacement

$\psi(z, it)$ by $\psi_\infty(z) = -2\sin\pi z$,
$\chi(z, it) = \psi'(z, it)/\psi(z, it)$ by $\chi_\infty(z) = \psi'_\infty(z)/\psi_\infty(z) = \pi\cot\pi z$,
$\wp(z, it) = \text{const} - \chi'(z, it)$ (explain) by $\wp_\infty(z) = \text{const} - \chi'_\infty(z) = \text{const} + \pi^2/\sin^2\pi z$.

Verify that in order that $\wp_\infty(z) = 1/z^2 + z^2 \cdot$ (holomorphic function near zero), we must have $\wp_\infty(z) = \dfrac{\pi^2}{\sin^2\pi z} - \dfrac{\pi^2}{3}$.

Verify also that $\wp_\infty$ satisfies the differential equation for $\wp$ for suitable values of E_4 and E_6 (find these values!).

4. Denote by e_1, e_2, e_3 the values $\wp(1/2), \wp(\tau/2), \wp((1 + \tau)/2)$ of $\wp$ at the half-periods.

(i) Show that $e_1 + e_2 + e_3 = 0$. Obtain expressions for $e_1^2 + e_2^2 + e_3^2$ and $e_1^3 + e_2^3 + e_3^3$ in terms of the coefficients g_2, g_3 of the differential equation $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$.

(ii)* Show that e_1, e_2, e_3 are pair-wise distinct. [Hint: the zeros of $\wp'$.]

5. Prove the addition theorem for $\wp$,

$$\begin{vmatrix} 1 & 1 & 1 \\ \wp(u) & \wp(v) & \wp(w) \\ \wp'(u) & \wp'(v) & \wp'(w) \end{vmatrix} = 0$$

if and only if two of u, v, w are congruent modulo Λ or $u + v + w \in \Lambda$.

[Hint: consider the determinant as a function of u with parameters v, w. The case $v + w \in \Lambda$ is exceptional (why?).]

6. Show that any holomorphic map f of degree 2 from an elliptic curve $\mathbb{C}/\Lambda$ to S^2 is given by a 'Möbius transformation of a shifted $\wp$-function':

$$f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},$$

for some $a, b, c, d, z_0 \in \mathbb{C}$.

7. Show, by considering the unit disc Δ and the complex plane $\mathbb{C}$, that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic).
Show that no two of the following domains in $\mathbb{C}$ are conformally equivalent

$$\{1 < |z| < 2\}, \qquad \{0 < |z| < 1\}, \qquad \{0 < |z| < \infty\}.$$

8. (i) Let R and S be some Riemann surfaces, $f : R \to S$ a continuous map, and p a point in R. Show, directly from the definition of holomorphic maps, that if f is holomorphic on $R \setminus \{p\}$ then f is in fact holomorphic on all of R.

(ii) Suppose that each of $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of four distinct points in S^2 and $F : S^2 \setminus A \to S^2 \setminus B$ is a biholomorphic map. Show that F extends to a biholomorphic map of S^2 onto itself, hence the β_4 is constrained to be in a finite subset of S^2 determined by the other β_i's and α_j's.

9. Show that if R and S are Riemann surfaces such that both are connected, R is compact and S is **non-compact** then every holomorphic map $f : R \to S$ is constant.

10. (i) Let R and S be compact connected Riemann surfaces and $g : R \to S$ a non-constant holomorphic map. Show that the genus of R is greater or equal to the genus of S.

(ii) Let R and S be compact connected Riemann surfaces, such that

$$\text{genus}(R) = \text{genus}(S) = g.$$

Show that if $f : R \to S$ is a non-constant holomorphic map and $g > 1$ then f is biholomorphic. What does the argument give in the case when (a) $g = 0$ or (b) $g = 1$?

(iii) Show that a holomorphic map $f : S^2 \to S^2$ of degree $k \geq 2$ must have ramification points (i.e. points $p \in S^2$ with $v_f(p) > 1$); recover from this the answer to Q7 in ex. sheet 1.

11. (i) Let f and g be two elliptic functions (with the same lattice of periods) and N a positive integer. By considering the poles of f and g, estimate from above the dimension of the complex vector space spanned by $f(z)^m g(z)^n$, for $0 \leq m, n \leq N$. Deduce that when N is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \qquad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions f, g on an elliptic curve $\mathbb{C}/\Lambda$ are **'algebraically related'**: there is a polynomial Q in two variables, so that $Q(f(z), g(z)) = 0$ for all z.

(ii)* Show that in fact (i) holds for meromorphic functions on **any compact** Riemann surface.

12. Recall from the Lectures that $\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$, where $\mathbf{e}(z) = \exp(2\pi i z)$ and $\text{Im}(\tau) > 0$. Show that if k is a positive integer then $\vartheta(0, \tau)^k = \sum_{n=0}^{\infty} r_n(k) e^{\pi n i \tau}$, where $r_n(k)$ is the number of ways to express the integer n as a sum of k squares.

Supervisors can obtain an annotated version of this example sheet from DPMMS.