Part IID RIEMANN SURFACES (2005–2006): Example Sheet 3

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1. Show that for any non-constant complex polynomial $P(s,t)$, the set $\{(s,t) \in \mathbb{C}^2 : P(s,t) = 0\}$ is unbounded. (I.e. any algebraic curve in $\mathbb{C}^2$ is non-compact.)

2. Let $S_0 = \{(s,t) \in \mathbb{C}^2 : t^2 = s^2 - a^2\}$, where a is a fixed non-zero complex number. Show that S_0 is a non-singular curve.

By finding the intersection point(s) of S_0 with the complex line $\lambda(s-a) = t$, show that the map $\varphi : \mathbb{C} \setminus \{1, -1\} \to S_0 \setminus \{(a,0)\}$ given by

$$\varphi(\lambda) = \left(a\frac{\lambda^2+1}{\lambda^2-1}, \frac{2a\lambda}{\lambda^2-1} \right)$$

is biholomorphic. Thus φ can be thought of as a 'parameterization' of an open subset in S_0.

3. Identify the non-singular projective curve S so that S_0 is biholomorphic to an open subset $\{X : Y : Z \in S \mid X \neq 0\}$ and write down the points of $S \setminus S_0$ (the points at 'infinity' of S).

Show that φ (as defined in Question 2) extends to a holomorphic map $\varphi : \mathbb{C} \to S$. Determine $\varphi(z)$, for $z = \pm 1$, and $\varphi(\mathbb{C})$.

Finally, identify $\mathbb{C}$ as $\mathbb{P}^1 - \{1 : 0\}$ via $\lambda \mapsto \lambda : 1$ and show, by further extending φ, that S is biholomorphic to the Riemann sphere.
[Hint: verifying that φ extends continuously to $\mathbb{P}^1$ will suffice — explain why.]

4. (Projective transformations.) Show that any linear isomorphism $A \in GL(3,\mathbb{C})$ induces a homeomorphism (still to be denoted by A) of the projective plane $\mathbb{P}^2$ onto itself. When do $A, B \in GL(3,\mathbb{C})$ induce the same map on $\mathbb{P}^2$?

Show further that the restriction of A to any non-singular projective curve $C \subset \mathbb{P}^2$ gives a biholomorphic map of C onto the image $A(C)$.

5. Let $E = \mathbb{C}/\Lambda$ be the elliptic curve defined by a lattice Λ and write $E_0 = E \setminus \{\Lambda\}$ for the complement of the coset of Λ. Show that

$$\Phi : z + \Lambda \in E_0 \to (\wp(z), \wp'(z)) \in \mathbb{C}^2$$

maps the punctured elliptic curve E_0 biholomorphically onto a non-singular algebraic curve in $\mathbb{C}^2$.

Show further that Φ extends to a biholomorphic map of E onto a non-singular projective curve $\{P(X,Y,Z) = 0\}$, for a certain homogeneous cubic polynomial P.
[Hint: the differential equation for $\wp$.]

6. (Hyperelliptic involution.) A compact Riemann surface S is called **hyperelliptic** if it admits a meromorphic function $f : S \to \mathbb{P}^1$ of degree 2. Show that, for any hyperelliptic Riemann surface S, the map $a : S \to S$ determined (uniquely) by the properties $f \circ a = f$, and $a(x) \neq x$ if $v_f(x) = 1$, is holomorphic.

7. Consider the complex algebraic curve C in $\mathbb{C}^2$ defined by the vanishing of the polynomial $P(s,t) = t^3 - s(s^2-1)$. Show that C is non-singular and find the branch locus of the branched cover $f : C \to \mathbb{C}$ given by the first projection. Find also the ramification points of f and the branching orders.

8. Analyze the compactification of the curve C of Question 7 along the following lines.

(i) For $|z| > 1$, show that there exists a holomorphic function $h(z)$ such that $h(z)^3 = 1 - z^{-2}$, $h(z) \to 1$ as $|z| \to \infty$.

(ii) Deduce, by writing the equation for C in the form $t^3 = (s \cdot h(s))^3$, that $C \cap \{(s,t) \in \mathbb{C}^2 : |s| > 1\} = C_1 \cup C_2 \cup C_3$, where the C_j are pairwise disjoint and the restriction of f to C_j gives a biholomorphic map to $\{s \in \mathbb{C} : |s| > 1\}$.

(iii) Hence show that there exists a compact Riemann surface $R = C \cup \{\infty_1\} \cup \{\infty_2\} \cup \{\infty_3\}$ together with a holomorphic map $F : R \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, such that the restriction of F to C is f and $F(\infty_j) = \infty$ $(j = 1, 2, 3)$.

Now find the genus of the surface R.

9. Prove that two divisors on the Riemann sphere are linearly equivalent if and only if they have the same degree.

10. Let D be an effective divisor on $\mathbb{P}^1$. Show, directly from the definition of ℓ, that $\ell(D) = \deg D + 1$.

11. Suppose that C is a non-singular irreducible complex projective curve and P a point in C with $\ell(P) > 1$. If $f \in \mathscr{L}(P)$ is non-constant, show that the map $\alpha : x \in C \to f(x) : 1 \in \mathbb{P}^1$ is biholomorphic. ($\alpha(x) = 1 : 0$ if x is a pole of f.) Show further that if D is an effective non-zero divisor on a non-singular irreducible complex projective curve not biholomorphic to $\mathbb{P}^1$ then $\ell(D) \leq \deg D$.

12. Let φ and ψ denote the charts on the Riemann sphere S^2 defined by the stereographic projections from, respectively, the North and South poles.

(i) Regarding the differential dz on $\mathbb{C}$ as a local expression for a differential on S^2 with respect to φ, determine the corresponding local expression for this differential with respect to ψ. Hence show that dz extends to a meromorphic differential on the Riemann sphere and has a pole at 'infinity'. Find the order of this pole.

(ii) More generally, if $(z-a)^n dz$ ($a \in \mathbb{C}$, $n \in \mathbb{Z}$) is a formula for a meromorphic differential $\omega_{a,n}$ on S^2 relative to φ give the formula for this differential relative to ψ. Write down the divisor of $\omega_{a,n}$.

(iii) Show also that there are no non-zero holomorphic differentials on S^2.

Does the holomorphic differential $e^z dz$ on $\mathbb{C}$ extend to a meromorphic differential on the Riemann sphere? Justify your answer.

13. Show that the differential dz on $\mathbb{C}$ induces a well-defined **holomorphic** differential η on an elliptic curve $\mathbb{C}/\Lambda$, via the standard charts given by local inverses of the quotient map $\mathbb{C} \to \mathbb{C}/\Lambda$. Deduce that the differential η is **not** obtainable as df for any meromorphic function f on $\mathbb{C}/\Lambda$.

Find a pair of meromorphic functions f and g on $\mathbb{C}/\Lambda$, so that $\eta = f dg$.

14.* Verify the Riemann–Roch theorem for the special cases of the Riemann sphere and an elliptic curve. You may assume results of any of the above questions.