Part IID RIEMANN SURFACES (2005–2006): Revision Example Sheet

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Note. The present 24-lecture course on Riemann Surfaces was first lectured in the academic year 2004–5. Thus the 2005 exam is the only one which was set on the present schedule and is readily suitable for revision.

The IID Riemann surface course was developed from the old 16-lecture course with the same title by incorporating some topics from Algebraic Curves. Most of the past exam questions on IIB Riemann surfaces are, in principle, suitable for revision (although elliptic curves used to be covered differently in some years). In fact, many of the questions on this sheet are compiled (and sometimes slightly modified) from those from past IIB papers. Most of the past exam questions on Algebraic Curves are *not* quite compatible with the present course (not before some editing, anyway).

It is never a bad idea to use for revision some of those questions from Sheets 1–4 (particularly the non-* ones) for which you did not have enough time during the term.

0. Attempt the four Riemann Surfaces questions from the 2005 exam (unless you prefer to save some of these and use as a 'mock exam').

1. Let $\varphi : S^2 \setminus \{N\} \to \mathbb{C}$ denote the stereographic projection from the North pole. Let U be an open region in $\mathbb{C}$. Show that a map $f : U \to S^2$ is holomorphic if and only if the function $\varphi \circ f$ is meromorphic on U and describe the poles of $\varphi \circ f$ in terms of f.

Deduce from the Weierstrass–Casorati Theorem that a biholomorphic map f of $\mathbb{C}$ onto itself cannot have an essential singularity at infinity. Hence determine the most general form of such f.

2. Let P and Q be complex polynomials of degree $m \geq 2$ with no common roots. Explain briefly how the rational function $P(z)/Q(z)$ induces a holomorphic map F from the 2-sphere $S^2 \cong \mathbb{C} \cup \{\infty\}$ to itself. What is the degree of F?

Let f and g be two holomorphic maps on the Riemann sphere $\mathbb{C} \cup \infty$. Show that

$$|\deg f - \deg g| \leq \deg(f + g) \leq \deg f + \deg g.$$

3. Let Λ be a lattice in $\mathbb{C}$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, where $\operatorname{Im} \tau > 0$. Describe briefly a construction of a family of charts making an elliptic curve $\mathbb{C}/\Lambda$ into a Riemann surface, so that the quotient map $\pi : z \in \mathbb{C} \to z + \Lambda \in \mathbb{C}/\Lambda$ is a holomorphic map.

Let f be an elliptic function with respect to Λ and suppose that f has degree d. Let A be the set of points $c \in \mathbb{C} \cup \{\infty\}$ for which the equation $f(u) = c$ has *less* than d distinct solutions (cosets) $u + \Lambda$. Prove that A is a non-empty finite set.

4. Let $h(z)$ be a Λ-periodic meromorphic function, where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ is a lattice in $\mathbb{C}$, $\operatorname{Im} \tau > 0$. Suppose that $h(z)$ has at the points $\frac{1}{2}(m + n\tau)$, $m, n \in \mathbb{Z}$, simple zeros precisely when $m + n$ is odd, and simple poles precisely when $m + n$ is even, and no other zeros or poles. Show that $h(z)$ is a constant multiple of $\dfrac{\wp'(z)}{\wp(z) - \wp(\frac{1+\tau}{2})}$ Here $\wp$ denotes the Weierstrass π-function with respect to the lattice Λ.

[Standard properties of the doubly-periodic meromorphic functions may be assumed without proof if accurately stated, but is you use properties of the $\wp$-function these should be deduced from first principles.]

5. (i) Prove that there is a compact Riemann surface of any genus $g \geq 0$. You should state accurately any auxiliary results that you require.

(ii) There is a theorem, known as 'the degree-genus formula', which asserts that a non-singular projective curve in $\mathbb{P}^2$ defined by a homogeneous polynomial of degree d has genus $(d-1)(d-2)/2$. In the lectures we considered an algebraic curve in $\mathbb{C}^2$ defined by a polynomial $y^2 - x^d - a_{d-1}x^{d-1} - \ldots - a_0$ of degree $d = 2g + 2$, then compactified this curve and found that the resulting compact surface has genus $g = (d-2)/2$. Does this give a counterexample to the degree-genus formula?

6. Let f, g be two meromorphic functions on a Riemann surface. Explain what is meant by a meromorphic differential $f dg$, and by a zero and a pole of $f dg$.

Let C_0 be a curve in $\mathbb{C}^2$ defined by the vanishing of polynomial $p(s,t) = t^2 - t - s^3$. Write out the equation of the corresponding projective curve in $C \subset \mathbb{P}^2$ and show that C is non-singular. Let $f_1 : (s,t) \in C_0 \to s \in \mathbb{C}$ and $f_2 : (s,t) \in C_0 \to t \in \mathbb{C}$ denote the restrictions to C_0 of the first and the second projection. Show that $\omega = \dfrac{df_1}{2f_2 - 1}$ defines a holomorphic differential on C_0. Does ω have zeros on C_0?

7. In this question S is a compact connected Riemann surface and you may assume that S admits non-constant meromorphic functions.

Define *canonical divisors* on a compact connected Riemann surface S and show that any two canonical divisors are linearly equivalent. State the Riemann–Roch theorem and deduce from it that the degree of a canonical divisor on S is determined by the topological surface underlying S.

Show that if S has genus 0 then S is biholomorphic to the Riemann sphere.

8. Let $\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$. Suppose that f is a power series that converges in some neighbourhood of the point 1 in $\mathbb{C}^*$, and that f can be analytically continued over every curve in $\mathbb{C}^*$ that starts at the point 1. Let $f_1(z)$ be a power series that converges in some neighbourhood of the point 1, and that is obtained by the analytic continuation of f around the circle $e^{2\pi i t}$, $0 \leq t \leq 1$. Show that if $f_1(z) = f(z)$ then f is single-valued on $\mathbb{C}^*$.

Describe how f determines a simply-periodic function F in the complex plane $\mathbb{C}$. What is F when $f(z) = z^k$, where k is a positive integer?

[Any form of the Monodromy Theorem may be used without proof provided that it is carefully stated. Elementary properties of homotopy may be used without proof. You might like to consider the universal covering $z \mapsto \exp(z)$ of $\mathbb{C}^*$.]