Part IID RIEMANN SURFACES (2004–2005): Example Sheet 4

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1. Suppose that a holomorphic function f satisfies a linear differential equation

$$f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \ldots + a_1(z)f'(z) + a_0(z)f(z) = 0$$

on an open domain $D \subset \mathbb{C}$, where $a_i(z)$ are holomorphic on $\mathbb{C}$. Show that every analytic continuation of (f, D) also satisfies this equation.

2. Prove that the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \ldots,$$

converges if $|z| < 1$ and diverges if $|z| > 1$. Further, prove that if $\varphi = p/2^q$ $(p, q \in \mathbb{Z})$, and $r > 0$ then $\lim_{r\to 1-} f(re^{i\pi\varphi}) = \infty$. Deduce that the unit circle is the natural boundary for the function element $(f, \{|z| < 1\})$.

3. (i) Prove Schwartz lemma: if $f : \Delta \to \Delta$ is holomorphic and $f(0) = 0$ then either $|f(z)| < |z|$, for every $z \in \Delta$, or $f(z) = e^{i\theta}z$, for some real θ. Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. [Hint: consider the function $g(z) = f(z)/z$ and apply the *maximum modulus principle* to $g(z)$ on the closed discs $\{|z| \le 1 - \epsilon\}$, for any small $\epsilon > 0$.]

(ii) Deduce from Schwartz lemma that any biholomorphic map of Δ onto itself is a Möbius transformation (restricted to Δ). You may assume without proof a result (from IB Geometry examples) that a Möbius transformation maps Δ onto itself if and only if it is of the form $z \mapsto \dfrac{az + \bar{c}}{cz + \bar{a}}$, with $|a|^2 - |c|^2 = 1$.
[Hint: reduce the problem to the case when a biholomorphic map of Δ onto itself has a fixed point $z = 0$.]

(iii) The group $SU(1, 1)$ is defined as the group of complex 2×2 matrices preserving the standard Hermitian form of signature $(1, 1)$ on $\mathbb{C}^2$, i.e.

$$SU(1,1) = \left\{A \in GL(2, \mathbb{C}) \ : \ \det A = 1 \text{ and } A\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\overline{A^t} = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\}.$$

Show that the group $\operatorname{Aut}\Delta$ of biholomorphic automorphisms of the open unit disc Δ is isomorphic to a 'projective special unitary group' $PSU(1, 1) = SU(1, 1)/\pm 1$.
(Compare with Q6 of example sheet 1.)

4. For α, $\beta \in \mathbb{C}$, show that the area of the parallelogram with vertices 0, α, β, $\alpha + \beta$ is $|\operatorname{Im}(\alpha\bar{\beta})|$. Show that two pairs α, β and λ, μ of complex numbers span the same lattice if and only if

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}k & l\\ m & n\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

for some **integers** k, l, m, n, with $kn - lm = \pm 1$.

5. A group Γ acts **properly discontinuously** on a topological space X if and only if every $x \in X$ has a neighbourhood U, so that the sets $\gamma(U)$, for all $\gamma \in \Gamma$, are disjoint. Assuming the results of Q6(ii) of Example sheet 1, prove that any subgroup of biholomorphic automorphisms of $\mathbb{C}$ acting properly discontinuously is one of the following groups of translations,

$$\text{(i) } \{0\}, \qquad \text{(ii) } \mathbb{Z}\omega, \quad \omega \in \mathbb{C}^*, \qquad \text{or (iii) } \mathbb{Z}\lambda + \mathbb{Z}\mu, \quad \lambda\mu \in \mathbb{C}, \lambda\bar{\mu} \notin \mathbb{R}.$$

Deduce that the only Riemann surfaces whose universal cover is $\mathbb{C}$ are $\mathbb{C}$ itself, $\mathbb{C}^*$, and the elliptic curves.

6. Show, using the uniformization theorem, that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than 1 is constant.

7. (The j-invariant.) (a) The cross-ratio of four distinct points is defined by $\lambda = (z, z_1; z_2, z_3) = (z_0 - z_1)(z_2 - z_3)/((z_1 - z_2)(z_3 - z_0))$. Extend this definition to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, by taking the limit if some $z_k = \infty$, and verify that λ can take any complex value except 0, 1 and ∞. Show also that the only values of the cross-ratio obtainable from the same four points taken in some order are λ, $1/\lambda$, $1-\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$, and $(\lambda-1)/\lambda$.

(b) Let $\varphi(\lambda) = 4(\lambda^2 - \lambda + 1)^3/(27\lambda^2(\lambda-1)^2)$. Show that two unordered quadruples are related by a Möbius transformation if (and only if) their cross-ratios λ, λ' satisfy $\varphi(\lambda) = \varphi(\lambda')$.

(c) In the lectures we saw that an elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is determined, up to isomorphism, by the values of Weierstrass function $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_2 = \wp(1/2 + \tau/2)$. For $\mathrm{Im}(\tau) > 0$, define $\lambda(\tau) = (e_1, e_2; e_3, \infty) = (e_1 - e_2)/(e_3 - e_2)$ and $J(\tau) = \varphi(\lambda(\tau))$. Show that $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is biholomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau')$ if and only if $J(\tau) = J(\tau')$ (Thus $J(\tau)$ parameterises uniquely the equivalence classes of biholomorphic elliptic curves.)

8. (Analytic continuation by reflections.) Let f be a function which is holomorphic on the upper half-plane $\mathbb{H}$ and continuous on $\mathbb{H} \cup I$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $f(z) \in \mathbb{R}$ whenever $z \in I$. Prove that $f(z) = \overline{f(\bar{z})}$, for $\mathrm{Im}(z) < 0$, defines an analytic continuation of f to $\mathbb{C} \setminus (\mathbb{R} \setminus I)$.
[Hint: it is convenient to use Morera's theorem from Further Analysis. At some stage, consider a sequence of contours $\gamma_n(t)$, such that the γ_n's converge *uniformly with first derivatives* to a contour $\gamma(t)$ containing a subinterval of $I \subset \mathbb{R}$.]

Define, using Möbius transformations, the reflection in a circle in $\mathbb{R}^2$, generalising the reflections in straight lines. Now state carefully a general form of the principle of analytic continuation by reflections in lines or circles.

9. Consider the interior of hyperbolic triangle $T = \{z \in \mathbb{H} : 0 < \mathrm{Re}(z) < 1, |z - 1/2| > 1/2\}$ in the upper half-plane $\mathbb{H}$. Let μ be a conformal equivalence map from T onto the upper half-plane and such that $\lim_{z\to 0} \mu(z) = 0$, $\lim_{z\to 1} \mu(z) = 1$, $\lim_{z\to\infty} \mu(z) = \infty$. (We assume the existence of such μ without proof here; it is a consequence of the Riemann mapping theorem. In fact, it is possible to give, with some further work, an 'explicit' construction of μ.) Assume further that μ extends continuously to the sides of the triangle T.

Show the following.

(a) μ has a well-defined analytic continuation, by reflections in the sides of T. By repeating the reflections in the boundary arcs sufficiently many times, one obtains an analytic continuation of μ defined at any point of $\mathbb{H}$.

(b) The resulting holomorphic function on $\mathbb{H}$ (still denoted by μ) does not take values 0 and 1.

(c) μ admits no further analytic continuation outside $\mathbb{H}$.

(d) μ realizes $\mathbb{H}$ as the universal covering space of $\mathbb{C} \setminus \{0, 1\}$.

10.* (Four views on the elliptic curves.) Let E be a compact connected Riemann surface. Show that the following are equivalent.

(1) E is the quotient $\mathbb{C}/\Lambda$ of the complex plane by a lattice.

(2) E is biholomorphic to a non-singular curve in $\mathbb{P}^2$ defined as the zero locus of a homogeneous cubic polynomial in the generalized Weierstrass normal form $XZ^2 - 4Y^3 - AX^2Y - BX^3$, for some complex constants A, B, $A^3 - 27B^2 \neq 0$.

(3) E is a compact Riemann surface of genus 1.

(4) there is a $2:1$ covering $E \to \mathbb{P}^1$ branched over four points.

You may assume without proof that any abelian discrete subgroup of $\mathrm{Aut}(\Delta) = SU(1,1)/\pm 1$ is a cyclic (this, and some topology, will be useful when showing that (3) implies (1)).

Supervisors can obtain an annotated version of this example sheet from DPMMS.