## Part IID RIEMANN SURFACES (2004-2005): Example Sheet 3

> (a.g.kovalev@dpmms.cam.ac.uk)

1. Let $S_{0}=\left\{(s, t) \in \mathbb{C}^{2}: t^{2}=s^{2}-a^{2}\right\}$, where $a$ is a fixed non-zero complex number. Show that $S_{0}$ is a non-singular curve.

By finding the intersection point(s) of $S_{0}$ with the complex line $\lambda(s-a)=t$, show that the $\operatorname{map} \varphi: \mathbb{C} \backslash\{1,-1\} \rightarrow S_{0} \backslash\{(a, 0)\}$ given by

$$
\varphi(\lambda)=\left(a \frac{\lambda^{2}+1}{\lambda^{2}-1}, \frac{2 a \lambda}{\lambda^{2}-1}\right)
$$

is biholomorphic.
2. Identify the non-singular projective curve $S$ so that $S_{0}$ is biholomorphic to an open subset $\{X: Y: Z \in S \mid X \neq 0\}$ and write down the points of $S \backslash S_{0}$ (at 'infinity' of $S$ ).
Show that $\varphi$ (as in Question 1) extends to a holomorphic map $\varphi: \mathbb{C} \rightarrow S$. Determine $\varphi(z)$, for $z= \pm 1$, and $\varphi(\mathbb{C})$.
3. (Projective transformations.) Show that any linear isomorphism $A \in G L(3, \mathbb{C})$ induces a homeomorphism (still to be denoted by $A$ ) of the projective plane $\mathbb{P}^{2}$ onto itself. When do $A, B \in G L(3, \mathbb{C})$ induce the same map on $\mathbb{P}^{2}$ ?
Show further that the restriction of $A$ to any non-singular projective curve $C$ gives a biholomorphic map of $C$ onto the image $A(C)$.
4. Let $E=\mathbb{C} / \Lambda$ be the elliptic curve defined by a lattice $\Lambda$ and write $E_{0}=E \backslash\{\Lambda\}$ for the complement of the coset of $\Lambda$. Show that

$$
\Phi: z+\Lambda \in E_{0} \rightarrow\left(\wp(z), \wp^{\prime}(z)\right) \in \mathbb{C}^{2}
$$

maps the punctured elliptic curve $E_{0}$ biholomorphically onto a non-singular algebraic curve in $\mathbb{C}^{2}$.

Show further that $\Phi$ extends to a biholomorphic map of $E$ onto a non-singular projective curve $\{P(X, Y, Z)=0\}$ for a certain homogeneous cubic polynomial $P$.
[Hint: the differential equation for $\wp$.]
5. Let $C=\{P(X, Y, Z)=0\}$ be a non-singular curve in $\mathbb{P}^{2}$ ( $P$ is a homogeneous polynomial) and assume that $0: 0: 1 \notin C$. Then $f: X: Y: Z \in C \rightarrow X: Y \in \mathbb{P}^{1}$ is a well-defined holomorphic map. Show that if $a: b: c$ is a point in $C$ then $v_{f}(a: b: c)>1$ if and only if $(\partial P / \partial Z)(a, b, c)=0$.

Show further that $v_{f}(a: b: c)=n$ if and only if

$$
\frac{\partial P}{\partial Z}(a, b, c)=\ldots=\frac{\partial^{n-1} P}{\partial Z^{n-1}}(a, b, c)=0, \quad \frac{\partial^{n} P}{\partial Z^{n}}(a, b, c) \neq 0 .
$$

This gives a systematic way to compute $v_{f}(a: b: c)$ in this case.
[Hint: use a similar result for curves in $\mathbb{C}^{2}$ proved in the lectures.]
6. Consider the complex algebraic curve $C$ in $\mathbb{C}^{2}$ defined by the vanishing of the polynomial $P(s, t)=t^{3}-s\left(s^{2}-1\right)$. Show that $C$ is non-singular and find the branch locus of the branched cover $f: C \rightarrow \mathbb{C}$ given by the first projection. Find also the ramification points of $f$ and the branching orders.
7. Analyze the compactification of the curve $C$ of Question 6 along the following lines.
(i) For $|z|>1$, show that there exists a holomorphic function $h(z)$ such that $h(z)^{3}=1-z^{-2}$, $h(z) \rightarrow 1$ as $|z| \rightarrow \infty$.
(ii) Deduce, by writing the equation for $C$ in the form $t^{3}=(s \cdot h(s))^{3}$, that $C \cap\left\{(s, t) \in \mathbb{C}^{2}:|s|>1\right\}=C_{1} \cup C_{2} \cup C_{3}$, where the $C_{j}$ are pairwise disjoint and the restriction of $\Phi$ to $C_{j}$ gives a biholomorphic map to $\{s \in \mathbb{C}:|s|>1\}$.
(iii) Hence show that there exists a compact Riemann surface $R=C \cup\left\{\infty_{1}\right\} \cup\left\{\infty_{2}\right\} \cup\left\{\infty_{3}\right\}$ together with a holomorphic map $F: R \rightarrow \mathbb{P}^{1}=\mathbb{C} \cup\{\infty\}$, such that the restriction of $F$ to $C$ is $f$ and $F\left(\infty_{j}\right)=\infty(j=1,2,3)$.

Now find the genus of the surface $R$.
8. (Hyperelliptic involution.) A compact Riemann surface $S$ is called hyperelliptic if it admits a meromorphic function $f: S \rightarrow \mathbb{P}^{1}$ of degree 2 . Show that, for any hyperelliptic Riemann surface $S$, the map $a: S \rightarrow S$ determined (uniquely) by the properties $f \circ a=f$, and $a(x) \neq x$ if $v_{f}(x)=1$, is holomorphic.
9. (i) Show that the differential $d z$ on $\mathbb{C}$ extends to a meromorphic differential on the Riemann sphere and find the order of its pole at 'infinity'. Deduce that there are no non-zero holomorphic differentials on $S^{2}$.
(ii) Show that the differential $d z$ on $\mathbb{C}$ induces a well-defined holomorphic differential $\eta$ on an elliptic curve $\mathbb{C} / \Lambda$ (via the standard charts given by local inverses of the quotient map $\mathbb{C} \rightarrow \mathbb{C} / \Lambda$ ). Deduce that the differential $\eta$ is not obtainable as $d f$ for a meromorphic function $f$ on $\mathbb{C} / \Lambda$. Find a pair of meromorphic functions $f$ and $g$ on $\mathbb{C} / \Lambda$, so that $\eta=f d g$.
10. Let $D$ be an effective divisor on $\mathbb{P}^{1}$. Show that $\ell(D)=\operatorname{deg} D+1$.
11. Suppose that $C$ is a non-singular irreducible complex projective curve and $P$ a point in $C$ with $\ell(P)>1$. If $f \in \mathscr{L}(P)$ is non-constant, show that the map $\alpha: x \in C \rightarrow f(x): 1 \in \mathbb{P}^{1}$ is biholomorphic. $(\alpha(x)=1: 0$ if $x$ is a pole of $f$.) Show further that if $D$ is an effective non-zero divisor on a non-singular irreducible complex projective curve not biholomorphic to $\mathbb{P}^{1}$ then $\ell(D) \leq \operatorname{deg} D$.
12. A planar graph consists of a finite collection of points in $\mathbb{R}^{2}$, called vertices, and a finite collection of curves in $\mathbb{R}^{2}$, called edges. Every edge is a homeomorphic image of $[0,1]$ and joins two vertices. Any two edges meet only at their endpoints (vertices), if at all. The faces of the graph are connected components of the complement of the edges - excluding the outer, unbounded component. The graph is said to be connected if any two vertices are joined by a chain of edges.
Show that, for any connected graph, $V-E+F=1$, where $V, E, F$ are the numbers of vertices, edges and faces. What happens to this formula if a graph is not necessarily connected?
Now consider the pull-back image of the graph in $S^{2}$ via a stereographic projection. Show that if all the faces (including the closure of the image of the unbounded region) are convex then the graph is connected. (Here convex means that, for any two points in a face, the shortest arc of the great circle joining them is contained in this face, an arc of some great circle if antipodal points.) Hence deduce Euler's formula $V-E+F=2$, for any polygonal decomposition of the sphere with convex faces.

