Part IIB RIEMANN SURFACES (2004–2005): Example Sheet 2

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1. If f is a meromorphic doubly-periodic (i.e. elliptic) function of degree $k > 0$ show that f' is an elliptic function whose degree ℓ satisfies $k+1 \leq \ell \leq 2k$. Give examples to show that both bounds are attained.

Recall from example sheet 1: $\psi(z,\tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2}))$ and satisfies $\psi(z+1) = -\psi(z)$, $\psi(z+\tau) = -\mathbf{e}(-\frac{\tau}{2} - z)\psi(z)$, where $\mathbf{e}(z) = \exp(2\pi i z)$, $\psi(z) = -\psi(-z)$, and has unique zero 'modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$'.

2. (i) Prove that if $z, w \in \mathbb{C}$, then

$$\wp(z) - \wp(w) = -\psi'(0)^2 \frac{\psi(z-w)\psi(z+w)}{\psi(z)^2\psi(w)^2}.$$

[Hint: Regarding one of w,z as parameter, prove that each side is Λ-periodic in the other variable and has same zeros and poles. Get multiplicative constant by considering Laurent expansion at zero.]

(ii) Deduce that $\wp'(z) = -\psi'(0)^3 \dfrac{\psi(2z)}{\psi(z)^4}$ and recover from this formula the zeros of $\wp'$.

3. Let $\chi(z) = \psi'(z)/\psi(z)$. Differentiate 2(i) and interchange z and w to obtain:

$$\frac{1}{2}\,\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} = \chi(z+w) - \chi(z) - \chi(w).$$

Remark for readers of Ahlfors or Jones & Singerman: their σ and ζ are not quite the same as ψ and χ here, but for some constants A, B, $\sigma(z) = \exp(Az^2 + B)\psi(z)$, so $\zeta(z) = 2Az + \chi(z)$.

4.* (challenging but feasible) Prove the addition formula for $\wp$,

$$\wp(z+w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2.$$

[You will need to differentiate the formula in Q3 and use the differential equation satisfied by $\wp$ to eliminate $\wp''$. Note also that $\wp = a - \chi'$, for some constant $a \in \mathbb{C}$ (can you see why?).]

5.* Elliptic functions may be thought of as generalizations of trigonometric functions. To make this more precise, consider $\psi(z, it)$ for $t > 0$. Show that for each fixed z,

$$\exp(\pi t/4)\psi(z, it) \to -2\sin(\pi z), \text{ as } t \to \infty.$$

This suggests the replacement

$\psi(z, it)$ by $\psi_\infty(z) = -2\sin \pi z$,
$\chi(z, it)$ by $\chi_\infty(z) = \psi'_\infty(z)/\psi_\infty(z) = \pi \cot \pi z$,
$\wp(z, it)$ by $\wp_\infty(z) = \text{const} - \chi'_\infty(z) = \text{const} + \pi^2/\sin^2 \pi z$.

Verify that in order that $\wp_\infty(z) = 1/z^2 + z^2 \cdot$ (holomorphic function near zero),
we must have $\wp_\infty(z) = \dfrac{\pi^2}{\sin^2 \pi z} - \dfrac{\pi^2}{3}$.

Verify also that $\wp_\infty$ satisfies the differential equation for $\wp$ for suitable values of E_4 and E_6 (find these values!).

6. Show that any holomorphic map f of degree 2 from an elliptic curve $\mathbb{C}/\Lambda$ to S^2 is given by a 'Möbius transformation of a shifted $\wp$-function':

$$f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},$$

for some $a, b, c, d, z_0 \in \mathbb{C}$.

7. Show, by considering the unit disc Δ and the complex plane $\mathbb{C}$, that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic).
Show that no two of the following domains are conformally equivalent

$$\{z : 1 < |z| < 2\}, \qquad \{z : 0 < |z| < 1\}, \qquad \{z : 0 < |z| < \infty\}.$$

8. (i) Let R and S be some Riemann surfaces, $f : R \to S$ a continuous map, and p a point in R. Show, directly from the definition of holomorphic maps, that if f is holomorphic on $R \setminus \{p\}$ then f is in fact holomorphic on all of R.

(ii) Suppose that each of $A = \{\alpha_1, \alpha_2, \alpha_2, \alpha_4\}$ and $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a set of four distinct points in S^2 and $F : S^2 \setminus A \to S^2 \setminus B$ is a biholomorphic map. Show that F extends to a biholomorphic map of S^2 onto itself, hence the β_4 is determined by the other β_i's and α_j's.

9. Show that if R and S are Riemann surfaces such that both are connected, R is compact and S is **non-compact** then every holomorphic map $f : R \to S$ is constant.

10. (i) Let R and S be compact connected Riemann surfaces and $g : R \to S$ a non-constant holomorphic map. Show that the genus of R is greater or equal to the genus of S.

(ii) Let R and S be compact connected Riemann surfaces, such that

$$\text{genus}(R) = \text{genus}(S) = g.$$

Show that if $f : R \to S$ is a non-constant holomorphic map and $g > 1$ then f is biholomorphic. What does the argument give in the case when (a) $g = 0$ or (b) $g = 1$?

(iii) Show that a holomorphic map $f : S^2 \to S^2$ of degree $k \geq 2$ must have ramification points (i.e. points $p \in S^2$ with $v_f(p) > 1$); recover from this fact the answer to Q7 in example sheet 1.

11. (i) Let f and g be two elliptic functions (same lattice of periods) and N a positive integer. By considering the poles of f and g, estimate from above the dimension of the complex vector space spanned by $f(z)^m g(z)^n$, for $0 \leq m, n \leq N$. Deduce that when N is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \qquad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions f, g on an elliptic curve $\mathbb{C}/\Lambda$ are 'algebraically related': there is a polynomial Q in two variables, so that $Q(f(z), g(z)) = 0$ for all z.

(ii)* Show that the argument of (i) works for meromorphic functions on **any compact** Riemann surface.

12. Recall from the Lectures that $\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$, where $\mathbf{e}(z) = \exp(2\pi i z)$ and $\text{Im}(\tau) > 0$. Show that if k is a positive integer then $\vartheta(0, \tau)^k = \sum_{n=0}^{\infty} r_n(k) e^{\pi n i \tau}$, where $r_n(k)$ is the number of ways to express the integer n as a sum of k squares.