Part IIB RIEMANN SURFACES (2004–2005): Example Sheet 1

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There is a partial overlap between some of the first few questions and the example sheets on IB Further Analysis given by Prof. Johnstone last year. This is intended as a refresher on Further Analysis, but any part that you have already done may of course be skipped now.

1. (i) If $f : D^*(a, r) \to \mathbb{C}$ is holomorphic and has a pole of order n at a, show that there exist $\varepsilon > 0$ and $R > 0$ such that for any given w with $|w| > R$, the equation $f(z) = w$ has exactly n distinct solutions z in $D^*(a, \varepsilon)$. (Here $D(a, r)^* = \{z \in \mathbb{C} : 0 < |z - a| < r\}$.)

(ii) What is the valency of $f(z) = \cos z$ at $z = 0$? Find explicitly the local conformal equivalence $\zeta(z)$ such that $f(z) = 1 + (\zeta(z))^2$. [Hint: recall the double-angle formulae.]

(iii) Suppose that f is holomorphic near the point a. Show that the valency of f at a is greater than 1 if and only if $f'(a) = 0$. More precisely, show that $v_f(a) = m$ if and only if

$$f^{(k)}(a) = 0 \text{ for } k = 1, \ldots, m-1, \quad f^{(m)}(a) \neq 0.$$

2. The following is a useful generalization of the argument principle. Let D be an open disc, γ a simple closed curve in D (oriented so that $n(\gamma, a) = 1$ for a inside γ), f meromorphic, g holomorphic on D, and γ does not pass through any zeros or poles of f. Then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = k_1 g(z_1) + \ldots + k_m g(z_m) - \ell_1 g(w_1) - \ldots - \ell_n g(w_n),$$

where z_j are the zeros of f inside γ, and w_j are the poles of f inside γ, and k_j and ℓ_j are, respectively, their orders.

Verify this result by factorizing f as in the proof of the standard argument principle (leave g alone).

3. Suppose that f is holomorphic on the closed disc $\overline{D(a, r)}$ and g is a **locally defined inverse** to f at a, i.e. for all w with $|w - f(a)| < \delta$, there is a **unique** $g(w)$ such that $f(g(w)) = w$. Prove that

$$g(w) = \frac{1}{2\pi i}\int_{\gamma(a,\varepsilon)} z\frac{f'(z)}{f(z) - w}dz,$$

where $\gamma(a, \varepsilon)$ is defined by $\gamma(a, \varepsilon)(t) = a + \varepsilon e^{it}$, $0 \le t \le 2\pi$ (with small radius ε).
[Hint: apply Q2.]

4. Show that

(i) $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$.

(ii)* $g(z) = \displaystyle\sum_{m,n=-\infty}^{\infty} \frac{1}{(z - n - mi)^3}$ is holomorphic on $\mathbb{C}\setminus\Lambda$, where $\Lambda = \{n + mi : m, n \in \mathbb{Z}\}$.

[Hint: Use the Weierstrass M-test from Analysis II to show that these series are locally uniformly convergent.]

5. (i) Show that a bounded holomorphic function on Δ^* extends holomorphically to all of Δ. (Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $\Delta^* = \Delta - \{0\}$.)

(ii) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and injective (1:1). Let $F : \Delta^* \to \mathbb{C}$ be determined by $F(z) = f(1/z)$. By considering $w \in f(\Delta)$ and using the Weierstrass–Casorati Theorem, prove that 0 is at worst a pole of F and therefore f extends holomorphically to S^2.

6. (i) Show that the group of Möbius transformations is isomorphic to $SL(2,\mathbb{C})/\pm 1$.

(ii) Assuming the results of Q5(ii), deduce that the group $\mathrm{Aut}(\mathbb{C})$ of biholomorphic maps of the complex plane onto itself consists of maps of the form $f(z) = az + b \ \ (a \neq 0)$.

7. Let $F : S^2 \to S^2$ be holomorphic and non-constant, with degree $d \geq 1$. Show that for all but a finite number of values $Q \in S^2$, the equation $F(P) = Q$ has d **distinct** solutions (for $P \in S^2$). When does $F(P) = Q$ have d distinct solutions for **every** Q?

8. If f is a rational map of degree d what are the possible degrees for its derivative f' ?

9. Find the Fourier series expansion for $\dfrac{1}{\sin 2\pi z}$ valid in the region $\{z : \mathrm{Im}\, z > 0\}$ and also Fourier series expansion valid in the region $\{z : \mathrm{Im}\, z < 0\}$.
[You may use any results you know about Laurent expansions.]

In the following questions, $\mathbf{e}(z) = \exp(2\pi i z)$ *and* $\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$, $\mathrm{Im}\tau > 0$ *(as in the Lectures). Notation* $\vartheta(z)$ *means that* τ *is fixed.*

10. Let $\varphi(x,t) = \vartheta(x, it)$. Show that φ satisfies the heat equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{4\pi}\frac{\partial^2 \varphi}{\partial x^2}$$

11. Let $\psi(z) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2}))$ (having fixed τ, with $\mathrm{Im}\,\tau > 0$). Show that

$$\psi(z+1) = -\psi(z), \qquad \psi(z+\tau) = -\mathbf{e}\Big(-\frac{\tau}{2} - z\Big)\psi(z)$$

and that

$$\psi(z) = -\psi(-z).$$

Deduce that $\psi(0) = 0$ and that $\psi(z) = 0$ if and only if $z = n + m\tau$ for some integers n and m. Prove also that

$$\vartheta\Big(z + \frac{1}{2} + \frac{\tau}{2}\Big) = -i\mathbf{e}\Big(-\frac{\tau}{8} - \frac{z}{2}\Big)\psi(z).$$

12. What is the residue at $\dfrac{1}{2} + \dfrac{\tau}{2}$ of $\dfrac{d}{dz}\log\vartheta = \dfrac{\vartheta'}{\vartheta}$? Show that if

$$f(z) = \frac{d}{dz}\log\vartheta(z-a)$$

then

$$f(z+1) = f(z), \qquad f(z+\tau) = f(z) - 2\pi i.$$

Deduce that if $\lambda_1, \ldots, \lambda_n$ and $a_1, \ldots, a_n$ are complex numbers then

$$\lambda_1 \frac{d}{dz}\log\vartheta(z-a_1) + \ldots + \lambda_n \frac{d}{dz}\log\vartheta(z-a_n)$$

is an elliptic meromorphic function if and only if $\lambda_1 + \ldots + \lambda_n = 0$. (This is yet another result analogous to the expansion of a rational function in partial fractions.)