Part IIB RIEMANN SURFACES (2003–2004): Example Sheet 3

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1. Suppose that a holomorphic function f satisfies some linear differential equation with constant coefficients on a domain $D \subset \mathbb{C}$. Show that every analytic continuation of (f, D) also satisfies this equation.

2. Recall from the Lectures that $\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$, where $\mathbf{e}(z) = \exp(2\pi i z)$ and $\mathrm{Im}(\tau) > 0$. Show that if k is a positive integer then $\vartheta(0,\tau)^k = \sum_{n=0}^{\infty} r_n(k) e^{\pi n i \tau}$, where $r_n(k)$ is the number of ways to express the integer n as a sum of k squares.

3. Show that any map f of degree 2 from the torus $\mathbb{C}/\Lambda$ to S^2 is given by a 'Möbius transformation of a shifted $\wp$-function':

$$f(z) = \frac{a\wp(z - z_0) + b}{c\wp(z - z_0) + d},$$

for some $a, b, c, d, z_0 \in \mathbb{C}$.

4. Prove that the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \dots,$$

converges if $|z| < 1$ and diverges if $|z| > 1$. Further, prove that if $\varphi = p/2^q$ $(p, q \in \mathbb{Z})$, and $r > 0$ then $\lim_{r\to 1-} f(re^{i\pi\varphi}) = \infty$. Deduce that the unit circle is the natural boundary for the function element $(f, \{|z| < 1\})$.

5. A **planar graph** consists of a finite collection of points in $\mathbb{R}^2$, called vertices, and a finite collection of curves in $\mathbb{R}^2$, called edges. Every edge is a homeomorphic image of $[0, 1]$ and joins two vertices. Any two edges meet only at their endpoints (vertices), if at all. The faces of the graph are connected components of the complement of the edges—excluding the outer, unbounded component. The graph is said to be **connected** if any two vertices are joined by a chain of edges.

Show that, for any connected graph, $V - E + F = 1$, where V, E, F are the numbers of vertices, edges and faces. What happens to this formula if a graph is not necessarily connected?

Now consider the pull-back image of the graph in S^2 via a stereographic projection. Show that if all the faces (*including* the closure of the image of the unbounded region) are **convex** then the graph is connected. (Here convex means that, for any two points in a face, the shortest arc of the great circle joining them is contained in this face, an arc of some great circle if antipodal points.) Hence deduce Euler's formula $V - E + F = 2$, for any polygonal decomposition of the sphere with convex faces.

6. For α, $\beta \in \mathbb{C}$, show that the area of the parallelogram with vertices 0, α, β, $\alpha + \beta$ is $|\,\mathrm{Im}(\alpha\bar{\beta})|$. Show that two pairs α, β and λ, μ of complex numbers span the same lattice if and only if

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

for some **integers** k, l, m, n, with $kn - lm = \pm 1$.

7. A group Γ acts **properly discontinuously** on a topological space X if and only if every $x \in X$ has a neighbourhood U, so that the sets $\gamma(U)$, for all $\gamma \in \Gamma$, are disjoint. Assuming the results of Q6(ii) of Example sheet 1, prove that any subgroup of biholomorphic automorphisms of $\mathbb{C}$ acting properly discontinuously is one of the following groups of translations,

$$\text{(i) } \{0\}, \qquad \text{(ii) } \mathbb{Z}\omega, \quad \omega \in \mathbb{C}^*, \qquad \text{or (iii) } \mathbb{Z}\lambda + \mathbb{Z}\mu, \quad \lambda\mu \in \mathbb{C}, \lambda\bar{\mu} \notin \mathbb{R}.$$

Deduce that the only Riemann surfaces whose universal cover is $\mathbb{C}$ are $\mathbb{C}$ itself, $\mathbb{C}^*$, and the compact surfaces of genus 1.

8. Show, using the uniformization theorem, that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than 1 is constant.

9. (The j-invariant.) (a) The cross-ratio of four distinct points is defined by $\lambda = (z, z_1; z_2, z_3) = (z_0 - z_1)(z_2 - z_3)/((z_1 - z_2)(z_3 - z_0))$. Extend this definition to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, by taking the limit if some $z_k = \infty$, and verify that λ can take any complex value except 0, 1 and ∞. Show also that the only values of the cross-ratio obtainable from the same four points taken in some order are λ, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, and $(\lambda - 1)/\lambda$.

(b) Let $\varphi(\lambda) = 4(\lambda^2 - \lambda + 1)^3/(27\lambda^2(\lambda - 1)^2)$. Show that two unordered quadruples are related by a Möbius transformation if (and only if) their cross-ratios λ, λ' satisfy $\varphi(\lambda) = \varphi(\lambda')$.

(c) In the lectures we saw that an elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is determined, up to isomorphism, by the values of Weierstrass function $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_2 = \wp(1/2 + \tau/2)$. For $\mathrm{Im}(\tau) \neq 0$, define $\lambda(\tau) = (e_1, e_2; e_3, \infty) = (e_1 - e_2)/(e_3 - e_2)$ and $J(\tau) = \varphi(\lambda(\tau))$. Show that $J(\tau)$ parameterises uniquely the isomorphism classes of compact Riemann surfaces of genus 1.

10. (Analytic continuation by reflections.) Let f be a function which is holomorphic on the upper half-plane $\mathbb{H}$ and continuous on $\mathbb{H} \cup I$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $f(z) \in \mathbb{R}$ whenever $z \in I$. Prove that $f(z) = \overline{f(\bar{z})}$, for $\mathrm{Im}(z) < 0$, defines an analytic continuation of f to $\mathbb{C} \setminus (\mathbb{R} \setminus I)$.
[Hint: it is convenient to use Morera's theorem from Further Analysis. At some stage, consider a sequence of contours $\gamma_n(t)$, such that both γ_n's and converge *uniformly with first derivatives* to a contour $\gamma(t)$ containing a subinterval of $I \subset \mathbb{R}$.]

Define, using Möbius transformations, the reflection in a circle in $\mathbb{R}^2$, generalising the reflections in straight lines. Now state carefully a general form of the principle of analytic continuation by reflections in lines or circles.

11. Consider the interior of hyperbolic triangle $T = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1, |z - 1/2| > 1/2\}$ in the upper half-plane $\mathbb{H}$. Let μ be a conformal equivalence map from T onto the upper half-plane and such that $\lim_{z\to 0} \mu(z) = 0$, $\lim_{z\to 1} \mu(z) = 1$, $\lim_{z\to\infty} \mu(z) = \infty$. (We assume the existence of such μ without proof here; it is a consequence of the Riemann mapping theorem. In fact, it is possible to give, with some further work, an 'explicit' construction of μ.)
Show the following.

(a) μ extends continuously to the sides of the triangle T and has a well-defined analytic continuation, by reflections in the sides of T. By repeating the reflections in the boundary arcs sufficiently many times, one obtains an analytic continuation of μ defined at any point of $\mathbb{H}$.

(b) The resulting holomorphic function on $\mathbb{H}$ (still denoted by μ) does not take values 0 and 1.

(c) μ admits no further analytic continuation outside $\mathbb{H}$.

(d) μ realizes $\mathbb{H}$ as the universal covering space of $\mathbb{C} \setminus \{0, 1\}$.