# Part IIB RIEMANN SURFACES (2003–2004): Example Sheet 2

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**1.** If $f$ is a meromorphic doubly-periodic (i.e. elliptic) function of degree $k > 0$ show that $f'$ is elliptic and that its degree $\ell$ satisfies $k + 1 \le \ell \le 2k$. Give examples to show that both bounds are attained.

Recall from example sheet 1: $\psi(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}(n + \frac{1}{2})^2\tau + (n + \frac{1}{2})(z + \frac{1}{2}))$ and satisfies $\psi(z + 1) = -\psi(z)$, $\psi(z + \tau) = -\mathbf{e}(-\frac{\tau}{2} - z)\psi(z)$, where $\mathbf{e}(z) = \exp(2\pi iz)$, $\psi(z) = -\psi(-z)$, and has unique zero 'modulo the lattice $\{n + \tau m\}$'.

**2.** (i) Prove that if $z, w \in \mathbb{C}$, then

$$\wp(z) - \wp(w) = -\psi'(0)^2 \frac{\psi(z - w)\psi(z + w)}{\psi(z)^2\psi(w)^2}.$$

[Hint: Regarding one of $w$,$z$ as parameter, prove that each side is $\Lambda$-periodic in the other variable and has same zeros and poles. Get multiplicative constant by considering Laurent expansion at zero.]

(ii) Deduce that $\wp'(z) = -\psi'(0)^3 \dfrac{\psi(2z)}{\psi(z)^4}$ and recover from this formula the zeros of $\wp'$.

**3.** Let $\chi(z) = \psi'(z)/\psi(z)$. Differentiate 2(i) and interchange $z$ and $w$ to obtain:

$$\frac{1}{2}\,\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} = \chi(z + w) - \chi(z) - \chi(w).$$

Remark for readers of Ahlfors or Jones & Singerman: their $\sigma$ and $\zeta$ are not quite the same as $\psi$ and $\chi$, but for some constants $A, B$, $\sigma(z) = \exp(Az^2 + B)\psi(z)$, so $\zeta(z) = 2Az + \chi(z)$.

**4.*** Prove the addition formula for $\wp$,

$$\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2.$$

[You will need to differentiate the formula in Q3 and use the differential equation satisfied by $\wp$ to eliminate $\wp''$.]

**5.*** Elliptic functions may be thought of as generalizations of trigonometric functions. To make this more precise, consider $\psi(z, it)$ for $t > 0$. Show that for each fixed $z$,

$$\exp(\pi t/4)\psi(z, it) \to -2\sin(\pi z), \text{ as } t \to \infty.$$

This suggests the replacement
$\psi(z, it)$ by $\psi_\infty(z) = -2\sin \pi z$,
$\chi(z, it)$ by $\chi_\infty(z) = \psi'_\infty(z)/\psi_\infty(z) = \pi \cot \pi z$,
$\wp(z, it)$ by $\wp_\infty(z) = \text{const} - \chi'_\infty(z) = \text{const} + \pi^2/\sin^2 \pi z$.

Verify that in order that $\wp_\infty(z) = 1/z^2 + z^2 \cdot$ (holomorphic function near zero), we must have $\wp_\infty(z) = \dfrac{\pi^2}{\sin^2 \pi z} - \dfrac{\pi^2}{3}$.

Verify also that $\wp_\infty$ satisfies the differential equation for $\wp$ for suitable values of $E_4$ and $E_6$ (find these values!).

**6.** Show, by considering the unit disc $\Delta$ and the complex plane $\mathbb{C}$, that homeomorphic Riemann surfaces need not be conformally equivalent (biholomorphic).
Show that no two of the following domains are conformally equivalent

$$\{z : 1 < |z| < 2\}, \qquad \{z : 0 < |z| < 1\}, \qquad \{z : 0 < |z| < \infty\}.$$

**7.** Show that if $R$ and $S$ are Riemann surfaces such that both are connected, $R$ is compact and $S$ is **non-compact** then every holomorphic map $f : R \to S$ is constant.

**8.** (i) Let $R$ and $S$ be compact connected Riemann surfaces and $g : R \to S$ a non-constant holomorphic map. Show that the genus of $R$ is greater or equal to the genus of $S$.

(ii) Let $R$ and $S$ be compact connected Riemann surfaces, such that

$$\text{genus}(R) = \text{genus}(S) = g.$$

Show that if $f : R \to S$ is a non-constant holomorphic map and $g > 1$ then $f$ is biholomorphic. What does the argument give in the case when (a) $g = 0$ or (b) $g = 1$?

(iii) Show that a holomorphic map $f : S^2 \to S^2$ of degree $k \geq 2$ must have ramification points (i.e. points $p \in S^2$ with $v_f(p) > 1$); recover from this fact the answer to Q7 in example sheet 1.

**9.** (i) Prove Schwartz lemma: if $f : \Delta \to \Delta$ is holomorphic and $f(0) = 0$ then either $|f(z)| < |z|$, for every $z \in \Delta$, or $f(z) = e^{i\theta}z$, for some real $\theta$. Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. [Hint: consider the function $g(z) = f(z)/z$ and apply the *maximum modulus principle* to $g(z)$ on the closed discs $\{|z| \leq 1 - \epsilon\}$, for any small $\epsilon > 0$.]

(ii) Deduce from Schwartz lemma that any biholomorphic map of $\Delta$ onto itself is a Möbius transformation (restricted to $\Delta$). You may assume without proof a result (from IB Geometry examples) that a Möbius transformation maps $\Delta$ onto itself if and only if it is of the form $z \mapsto \dfrac{az + \bar{c}}{cz + \bar{a}}$, with $|a|^2 - |c|^2 = 1$.
[Hint: reduce the problem to the case when a biholomorphic map of $\Delta$ onto itself has a fixed point $z = 0$.]

(iii) The group $SU(1,1)$ is defined as the group of complex $2 \times 2$ matrices preserving the standard Hermitian form of signature $(1,1)$ on $\mathbb{C}^2$, i.e.

$$SU(1,1) = \left\{ A \in GL(2,\mathbb{C}) \ : \ \det A = 1 \text{ and } A\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\overline{A^t} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}.$$

Show that the group $\operatorname{Aut}\Delta$ of biholomorphic automorphisms of the open unit disc $\Delta$ is isomorphic to a 'projective special unitary group' $PSU(1,1) = SU(1,1)/\pm 1$.
(Compare with Q6 of example sheet 1.)

**10.** (i) Let $f$ and $g$ be two elliptic functions (same lattice of periods) and $N$ a positive integer. By considering the poles of $f$ and $g$, estimate from above the dimension of the complex vector space spanned by $f(z)^m g(z)^n$, for $0 \leq m, n, \leq N$. Deduce that when $N$ is sufficiently large there must be a non-trivial linear dependence,

$$\sum_{m,n=0}^{N} a_{m,n} f(z)^m g(z)^n \equiv 0, \qquad \text{for some } a_{m,n} \in \mathbb{C}.$$

Hence show that any two meromorphic functions $f, g$ on an elliptic curve $\mathbb{C}/\Lambda$ are 'algebraically related': there is a polynomial $Q$ in two variables, so that $Q(f(z), g(z)) = 0$ for all $z$.

(ii)* show that the argument of (i) in fact works on **any compact** Riemann surface.