

Part IIB RIEMANN SURFACES (2003–2004): Example Sheet 1

(a.g.kovalev@dpmms.cam.ac.uk)

1. (i) If $f : D(a, r) \rightarrow \mathbb{C}$ is holomorphic and has valency n at a , show that $\frac{1}{f - f(a)}$ has a pole of order n at a . (Here $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.)

(ii) What is the valency of $f(z) = \cos z$ at $z = 0$? Find explicitly the local conformal equivalence $\zeta(z)$ such that $f(z) = 1 + (\zeta(z))^2$. [Hint: recall the double-angle formulae.]

(iii) Suppose that f is holomorphic near the point a . Show that the valency of f at a is greater than 1 if and only if $f'(a) = 0$. More precisely, show that $v_f(a) = m$ if and only if

$$f^{(k)}(a) = 0 \text{ for } k = 1, \dots, m-1, \quad f^{(m)}(a) \neq 0.$$

2. The following is a useful generalization of the argument principle. Let D be an open disc, γ a simple closed curve in D (oriented so that $n(\gamma, a) = 1$ for a inside γ), f meromorphic, g holomorphic on D , and γ does not pass through any zeros or poles of f . Then

$$\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} dz = k_1 g(z_1) + \dots + k_m g(z_m) - \ell_1 g(w_1) - \dots - \ell_n g(w_n),$$

where z_j are the zeros of f inside γ , and w_j are the poles of f inside γ , and k_j and ℓ_j are, respectively, their orders.

Verify this result by *factorizing* f as in the proof of the standard argument principle (leave g alone).

3. Suppose that f is holomorphic on the closed disc $\overline{D(a, r)}$ and g is a **locally defined inverse** to f at a , i.e. for all w with $|w - f(a)| < \delta$, there is a **unique** $g(w)$ such that $f(g(w)) = w$. Prove that

$$g(w) = \frac{1}{2\pi i} \int_{\gamma(a, \varepsilon)} z \frac{f'(z)}{f(z) - w} dz,$$

where $\gamma(a, \varepsilon)$ is defined by $\gamma(a, \varepsilon)(t) = a + \varepsilon e^{it}$, $0 \leq t \leq 2\pi$ (with small radius ε).

[Hint: apply Q2.]

4. Show that

$$(i) \quad f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \text{ is holomorphic on } \mathbb{C} \setminus \mathbb{Z}.$$

$$(ii)^* \quad g(z) = \sum_{m, n=-\infty}^{\infty} \frac{1}{(z-n-mi)^3} \text{ is holomorphic on } \mathbb{C} \setminus \Lambda, \text{ where } \Lambda = \{n+mi : m, n \in \mathbb{Z}\}.$$

[Hint: Use the Weierstrass M-test from Analysis II to show that these series are locally uniformly convergent.]

5. (i) Show that a bounded holomorphic function on Δ^* extends holomorphically to all of Δ . (Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $\Delta^* = \Delta - \{0\}$.)

(ii) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic and injective (1:1). Let $F : \Delta^* \rightarrow \mathbb{C}$ be determined by $F(z) = f(1/z)$. By considering $w \in f(\Delta)$ and using the Weierstrass–Casorati Theorem, prove that 0 is at worst a pole of F and therefore f extends holomorphically to S^2 .

6. (i) Show that the group of Möbius transformations is isomorphic to $SL(2, \mathbb{C})/\pm 1$.

(ii) Assuming the results of Q5(ii), deduce that the group $\text{Aut}(\mathbb{C})$ of biholomorphic maps of the complex plane onto itself consists of maps of the form $f(z) = az + b$ ($a \neq 0$).

7. Let $F : S^2 \rightarrow S^2$ be holomorphic and non-constant, with degree $d \geq 1$. Show that for all but a finite number of values $Q \in S^2$, the equation $F(P) = Q$ has d **distinct** solutions (for $P \in S^2$). When does $F(P) = Q$ have d distinct solutions for **every** Q ?

8. If f is a rational map of degree d what are the possible degrees for its derivative f' ?

9. Find the Fourier series expansion for $\frac{1}{\sin 2\pi z}$ valid in the region $\{z : \text{Im } z > 0\}$ and also Fourier series expansion valid in the region $\{z : \text{Im } z < 0\}$.
[You may use any results you know about Laurent expansions.]

In the following questions, $\mathbf{e}(z) = \exp(2\pi iz)$ and $\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$ (as in the Lectures).

10. Let $\varphi(x, t) = \vartheta(x, it)$. Show that φ satisfies the heat equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{4\pi} \frac{\partial^2 \varphi}{\partial x^2}$$

11. Let $\psi(z) = \sum_{n=-\infty}^{\infty} \mathbf{e}(\frac{1}{2}(n + \frac{1}{2})^2\tau + (n + \frac{1}{2})(z + \frac{1}{2}))$ (having fixed τ , with $\text{Im } \tau > 0$). Show that

$$\psi(z+1) = -\psi(z), \quad \psi(z+\tau) = -\mathbf{e}\left(-\frac{\tau}{2} - z\right)\psi(z)$$

and that

$$\psi(z) = -\psi(-z).$$

Deduce that $\psi(0) = 0$ and that $\psi(z) = 0$ if and only if $z = n + m\tau$ for some integers n and m . Prove also that

$$\vartheta\left(z + \frac{1}{2} + \frac{\tau}{2}\right) = -i\mathbf{e}\left(-\frac{\tau}{8} - \frac{z}{2}\right)\psi(z).$$

12. What is the residue at $\frac{1}{2} + \frac{\tau}{2}$ of $\frac{d}{dz} \log \vartheta = \frac{\vartheta'}{\vartheta}$? Show that if

$$f(z) = \frac{d}{dz} \log \vartheta(z - a)$$

then

$$f(z+1) = f(z), \quad f(z+\tau) = f(z) - 2\pi i.$$

Deduce that if $\lambda_1, \dots, \lambda_n$ and a_1, \dots, a_n are complex numbers then

$$\lambda_1 \frac{d}{dz} \log \vartheta(z - a_1) + \dots + \lambda_n \frac{d}{dz} \log \vartheta(z - a_n)$$

is an elliptic meromorphic function if and only if $\lambda_1 + \dots + \lambda_n = 0$. (This is yet another result analogous to the expansion of a rational function in partial fractions.)