Representation Theory — Examples Sheet 4

On this sheet all representations are complex representations unless stated otherwise.

1. (i) Let χ_n be the character of the irreducible representation of $SU(2)$ of dimension $n+1$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\chi_n\overline{\chi_m}\,\mathrm{d}\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = -\frac{1}{2}(z - z^{-1})^2$.

(ii) Deduce that $K(z)$ is the volume (=area!) with respect to Haar measure, of the conjugacy class of $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.

2. *Optional.* Write down a Haar integral on $SU(2)$ and prove that it is translation invariant and normalised correctly.

3. Let $G = SU(2)$ and V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y viewed as an irreducible representation of G.

   (a) Show that V_n is isomorphic to its dual V_n^* as representations of G.

   (b) Decompose the representations $V_4 \otimes V_3$, $V_3 \otimes V_3$, S^2V_3 and $\Lambda^2 V_3$ into irreducibles.

   (c) How do $V_1^{\otimes n}$, $S^n V_1$, $S^2 V_n$ and $\Lambda^2 V_n$ decompose into irreducibles for $n \geq 1$. What about $S^3 V_2$?

4. Let $SU(2)$ act on the space $M_3(\mathbb{C})$ of 3×3 complex matrices by

$$A\colon X \mapsto A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this defines a representation of $SU(2)$ and decompose it into irreducibles.

5. Let G be a compact group. Show that if G has an n-dimensional faithful representation over $\mathbb{R}$ then there is a faithful continuous group homomorphism from G to the orthogonal group $O(n)$.

6. (i) By considering the action of $SU(2)$ by conjugation on the vector space of 2×2 complex matrices A such that $A = -\overline{A}^T$ and $\operatorname{tr} A = 0$, equipped with norm $||A||^2 = \det A$, construct a continuous group homomorphism $SU(2) \to SO(3)$. Deduce that $SU(2)/\{\pm I\} \cong SO(3)$ as topological groups.

   (ii) Describe the irreducible continuous complex representations of $SO(3)$.

7. Describe all the irreducible continuous one dimensional representations of the group $\mathbb{C}^*$.

8. Let G be a finite group, and $H \leq G$ a subgroup. If W is a representation of G, and V a representation of H, show

$$Ind_H^G(W \otimes V) \simeq W \otimes Ind_H^G V$$

by directly constructing a isomorphism.

9. Let $G = SL_2(F_q)$, $q = p^n$, p a prime, $p \neq 2$, $n \geq 1$. [*You may assume $n = 1$ if you haven't taken Galois theory; there is no difference in statements or proofs.*] Let T be diagonal matrices in G, and B be upper triangular matrices. Fix $\epsilon \in F_q$ such ϵ is not a square, and let $T_{ns} = \{\begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix} \in G\}$ be a 'non-split torus'.

   Compute the character table of $SL_2(F_q)$, completing the outline of the proof sketched in class.

   More precisely, decompose the representations $Ind_B^G\theta$, when θ is a one dimensional representation of B. Write $Ind_B^G\mathbf{1} = \mathbf{1} \oplus St$. Let ψ be a one dimensional representation of T_{ns}. By considering $R(\psi) =: Ind_B^G\theta \otimes (St - \mathbf{1}) - Ind_{T_{ns}}^G\psi$, for any θ with $\theta(-1) = \psi(-1)$, or otherwise, find representations of G of dimension $q-1$, and determine when they are distinct or irreducible. If they are not irreducible, determine the characters of the irreducible summands.

   Check that you have everything, both by comparing with the number of conjugacy classes and by summing the squares of the dimensions of the irreducibles.

10. With notation as in the previous question, decompose $Ind_T^G \theta$ and $Ind_{T_{ns}}^G \psi$ as a direct sum of irreducible representations.

11. The *Heisenberg group* is the group G of order p^3 of upper unitriangular matrices over the field with p elements. Show that G has p conjugacy classes of size 1 and $p^2 - 1$ conjugacy classes of size p. Find p^2 characters of G of degree 1.

    Find an abelian subgroup H of G of order p^2. By induction of characters from H to G show that G has $p - 1$ irreducible characters of degree p. Write down the character table of G.

IG 2025