Representation Theory — Examples Sheet 3

On this sheet all groups are finite and all representations are complex representations

1. Let V, W be finite dimensional vector spaces over k. Choose bases for V and W, and show that under the resulting identification of $V \otimes W$ with matrices the image of $\otimes : V \times W \to V \otimes W$ consists of *precisely* the matrices of rank ≤ 1.

2. Calculate S^2V and $\Lambda^2 V$ for the two-dimensional irreducible representations of D_8 and of Q_8. Which has the trivial representation as a subrepresentation in each case?

3. Find an isomorphism from $k[X] \to k[X]^*$. Find an isomorphism between $k[H\backslash G] \to k[G/H]$ (right to left cosets).

4. List the cosets $S_{a+b}/(S_a \times S_b)$, and the double cosets $(S_a \times S_b)\backslash S_{a+b}/(S_a \times S_b)$.

5. Show both directly and using characters that if U, V and W are representations of G then

$$V^* \otimes W \cong \mathrm{Hom}_k(V, W) \text{ and } \mathrm{Hom}_k(V \otimes W, U) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(W, U))$$

as representations of G. Deduce that if V is self-dual then either S^2V or $\Lambda^2 V$ contains a non-zero subrepresentation with trivial G-action.

6. Suppose $\rho \colon G \to GL(V)$ is an irreducible representation of G with character χ. By considering $V \otimes V$, S^2V and $\Lambda^2 V$ show that

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \begin{cases} 0 & \text{if } \chi \text{ is not real-valued} \\ \pm 1 & \text{if } \chi \text{ is real valued.} \end{cases}$$

Deduce that if $|G|$ is odd then G has only one real-valued irreducible character.

7. Let $\rho \colon G \to GL(V)$ be a representation of G of dimension d.
   (a) Compute $\dim S^n V$ and $\dim \Lambda^n V$ for all n.
   (b) Let $g \in G$ and $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\rho(g)$. What are the eigenvalues of g on $S^n V$ and $\Lambda^n V$?
   (c) Let $f(t) = \det(tI - \rho(g))$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of f and $\chi_{\Lambda^n V}$?
   (d) What is the relationship between $\chi_{S^n V}(g)$ and f? (Hint: start with case $d = 1$).

8. Recall the character table of D_{10} from sheet 2. Explain how to view D_{10} as a subgroup of A_5 and then use induction from D_{10} to A_5 to reconstruct the character table of A_5.

9. Obtain the character table of the dihedral group D_{2m} by using induction from the cyclic group C_m. For each irreducible representation of C_m, deteremine whether the induced representation is irreducible, and determine its restriction to C_m. You will want to split into two cases according as m is odd or even.

10. Find all the characters of S_5 obtained by inducing irreducible representations of S_4. Use these to reconstruct the character table of S_5. Check that the answer that you get is compatible with Mackey's formula for the inner product of the induced representation with itself. Then repeat, replacing S_4 by the subgroup $\langle (12345), (2354) \rangle$ of S_5 of order 20.

11. Prove that if H is a subgroup of a group G, and K is a subgroup of H, and W is a representation of K then $\mathrm{Ind}_K^G W \cong \mathrm{Ind}_H^G \mathrm{Ind}_K^H W$.

12. Let H be a subgroup of a group G. Show that for every irreducible representation (ρ, V) of G there is an irreducible representation (σ, W) of H such that ρ is an irreducible component of $\mathrm{Ind}_H^G W$.

    Deduce that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G/A|$.

13. We showed in class that there was an isomorphism $\mathbb{C}G \to \bigoplus_{V \text{irred}} End\, V$ as representations of $G\times G$, by computing characters . Define this map 'naturally' (as a map of vector spaces), and show it is an *algebra* isomorphism.

14. Suppose that G is a Frobenius group with Frobenius kernel K. Show that if V is a non-trivial irreducible representation of K then $\mathrm{Ind}_K^G V$ is also irreducible. Hence, explain how to construct the character table of G given the character tables of K and G/K.

15. Let $G = GL_2(F_q)$, $q = p^n$, p a prime, $n \geq 1$. [*You may assume* $n = 1$ *if you haven't taken Galois theory; there is no difference in statements or proofs.*] Let T be diagonal matrices in G, and B be upper triangular matrices.

    Determine how many irreducible representations are in $Ind_B^G\theta$, when θ is a one dimensional representation of B, and when $Ind_B^G\theta$ and $\mathrm{Ind}_B^G\,\theta'$ have constituents in common.

    Determine the character of $Ind_B^G\theta$.

    Describe the restriction of $\mathrm{Ind}_B^G\,\theta$ to $SL_2(F_q)$, and for which θ the representation $\mathrm{Ind}_B^G\,\theta$ comes from a representation of PGL_2.

    ** If $\mathrm{Ind}_B^G\,\theta$ is not irreducible, determine the dimension of its irreducible constituents. Determine their character.

Optional exercise. Galois theory needed.

16. (i) All of the the theorems of class work for representations over $\overline{\mathbb{Q}}$, not just $\mathbb{C}$, with the same proofs. Deduce that if G is a finite group, there exists a finite Galois extension $F \supseteq \mathbb{Q}$ such that every representation of G can be realised as a vector space over F. [Hint: consider the matrices defining the irreducible representations.] In fact we can take $F \subseteq \mathbb{Q}(e^{2\pi i/|G|})$. [Can you prove this?]

    In particular, conclude the character values are in F.

    (ii) For each of D_8 and Q_8, find a minimal field containing $\mathbb{Q}$ that contains the matrix coefficients of the 2-dimensional representation.

    (iii) Let V be a complex representation of G. Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Prove that $g \mapsto \sigma\chi_V(g)$ is the character of a representation of G. Give examples where this differs from V.

    (iv) Show that the following are equivalent: a) the character values are all integers, b) $\chi_V(g) \in \mathbb{Q}$ for all irreducible representations V and all $g \in G$, and c) For all $g \in G$, m coprime to the order of g, and irreducible representation V, $\chi_V(g) = \chi_V(g^m)$.

    If this is the case, deduce that for all $g \in G$, $\varphi(ord(g)) \mid |G|$, where φ is Euler's totient function. In particular, $p - 1 \mid |G|$ if $p \mid |G|$.

    (Such groups are even rarer than this exercise suggests!).

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