## MATHEMATICAL TRIPOS, PART II, 2020/2021
## REPRESENTATION THEORY EXAMPLE SHEET 2

*Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field $F$ of characteristic zero, usually $\mathbb{C}$.*

**1** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$ of dimension $d$, and affording character $\chi$. Show that $\ker \rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leqslant d$ for all $g \in G$, with equality if and only if $\rho(g) = \lambda I$, a scalar multiple of the identity, for some root of unity $\lambda$.

**2** Let $\chi$ be the character of a representation $V$ of $G$ and let $g$ be an element of $G$. If $g$ is an involution (i.e. $g^2 = 1 \neq g$), show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If $G$ is simple (but not $C_2$), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: consider the determinant of $g$ acting on $V$.) If $g$ has order 3 and is conjugate to $g^{-1}$, show that $\chi(g) \equiv \chi(1) \bmod 3$.

**3** Construct the character table of the dihedral group $D_8$ and of the quaternion group $Q_8$. You should notice something interesting.

**4** Construct the character table of the dihedral group $D_{10}$.

Each irreducible representation of $D_{10}$ may be regarded as a representation of the cyclic subgroup $C_5$. Determine how each irreducible representation of $D_{10}$ decomposes into irreducible representations of $C_5$.

Repeat for $D_{12} \cong S_3 \times C_2$ and the cyclic subgroup $C_6$ of $D_{12}$.

**5** Construct the character tables of $A_4$, $S_4$, $S_5$, and $A_5$.

The group $S_n$ acts by conjugation on the set of elements of $A_n$. This induces an action on the set of conjugacy classes and on the set of irreducible characters of $A_n$. Describe the actions in the cases where $n = 4$ and $n = 5$.

**6** A certain group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters $\alpha$ and $\beta$. The table below gives the sizes of the conjugacy classes in the group and the values which $\alpha$ and $\beta$ take on them.

| | 1 | 15 | 40 | 90 | 45 | 120 | 144 | 120 | 90 | 15 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 6 | 2 | 0 | 0 | 2 | 2 | 1 | 1 | 0 | $-2$ | 3 |
| $\beta$ | 21 | 1 | $-3$ | $-1$ | 1 | 1 | 1 | 0 | $-1$ | $-3$ | 0 |

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

**7** The table below is a part of the character table of a certain finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

| | 1 | 3 | 3 | 7 | 7 |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | $\zeta$ | $\bar{\zeta}$ |
| $\chi_2$ | 3 | $\gamma$ | $\bar{\gamma}$ | 0 | 0 |

**8** Let $x$ be an element of order $n$ in a finite group $G$. Say, without detailed proof, why
(a) if $\chi$ is a character of $G$, then $\chi(x)$ is a sum of $n$th roots of unity;
(b) $\tau(x)$ is real for every character $\tau$ of $G$ if and only if $x$ is conjugate to $x^{-1}$;
(c) $x$ and $x^{-1}$ have the same number of conjugates in $G$.

Prove that the number of irreducible characters of $G$ which take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes (so-called *real classes*).

**9** A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters $\alpha$, $\beta$ and $\gamma$. The table below gives the sizes of the conjugacy classes and the values $\alpha$, $\beta$ and $\gamma$ take on them.

| | 1 | 21 | 42 | 56 | 24 | 24 |
|---|---|---|---|---|---|---|
| $\alpha$ | 14 | 2 | 0 | $-1$ | 0 | 0 |
| $\beta$ | 15 | $-1$ | $-1$ | 0 | 1 | 1 |
| $\gamma$ | 16 | 0 | 0 | $-2$ | 2 | 2 |

Construct the character table of the group.
[You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where $\zeta$ is a primitive 7th root of unity.]

The character table thus obtained is in fact the character table of the group $G = \mathrm{PSL}_2(7)$ of $2 \times 2$ matrices with determinant 1 over the field $\mathbb{F}_7$ (of seven elements) modulo the two scalar matrices. Deduce directly from the character table that $G$ is simple[1].

**10** While walking down King's Parade you find a scrap of paper with the following character table on it:

| | | | | |
|---|---|---|---|---|
| | 1 | | 1 | |
| | 1 | | $-1$ | |
| $\cdots$ | 2 | $\cdots$ | $-1$ | $\cdots$ |
| | 3 | | 1 | |
| | 3 | | $-1$ | |

All except two of the columns are obscured, and while it is clear that there are five rows, you cannot read anything of the other columns, including their position. Prove that there is an error in the table. Given that there is exactly one error, determine where it is, and what the correct entry should be.

**11** Let a finite group $G$ act on itself by conjugation. Find the character of the corresponding permutation representation.

[1] It is known that there are precisely five non-abelian simple groups of order less than 1000. The smallest of these is $A_5 \cong \mathrm{PSL}_2(5)$, while $G$ is the second smallest. The others are $A_6$, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(11)$. It is also known that for $p \geqslant 5$, $\mathrm{PSL}_2(p)$ is simple.

**12** Consider the character table $Z$ of $G$ as a matrix of complex numbers (as we did when deriving the column orthogonality relations from the row orthogonality relations).

(a) Using the fact that the complex conjugate of an irreducible character is also an irreducible character, show that the determinant $\det Z$ is $\pm \det \bar{Z}$, where $\bar{Z}$ is the complex conjugate of $Z$.

(b) Deduce that either $\det Z \in \mathbb{R}$ or $\det Z \in i\mathbb{R}$.

(c) Use the column orthogonality relations to calculate the product $\bar{Z}^T Z$, where $\bar{Z}^T$ is the transpose of the complex conjugate of $Z$.

(d) Calculate $|\det Z|$.

SM, Michaelmas Term 2020
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`