PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first seven questions we let $G = \mathrm{SU}(2)$. Questions 9–12 deal with a variety of topics at Tripos standard.

1 Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. What is its character? Show that V_n is isomorphic to its dual V_n^*.

2 Using the properties of exterior and symmetric powers, together with the Clebsch-Gordan formula, decompose the following spaces into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to the given G-space; you are not being asked to find such an isomorphism explicitly).

(i) $V_4 \otimes V_3$, $V_3^{\otimes 2}$, $\Lambda^2 V_3$;
(ii) $V_1^{\otimes n}$;
(iii) $S^2 V_n$, $\Lambda^2 V_n$ $(n \geqslant 1)$, $S^3 V_2$
(iv) $S^n V_1$ for $n \geqslant 1$.

3 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by conjugation:

$$A : \ X \ \mapsto \ A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducible summands.

4 Let χ_n be the character of the irreducible representation ρ_n of G on V_n of dimension $n + 1$.

Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

5 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines a G-invariant inner product on the vector space of integrable functions on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

6 Either of the following ways can be used to identify SO(3) with real projective 3-space $\mathbb{R}P^3$ (the topological space of lines passing through the origin in $\mathbb{R}^{n+1}$), and hence show that $G/\{\pm I_2\} \cong \mathrm{SO}(3)$.

(a) [Sketched in lectures: fill out the remaining details.] Let $\mathbb{H}^0 = \{ai + bj + ck : a, b, c \in \mathbb{R}\}$ be the 3-dimensional space of *pure quaternions*, and let the quaternions of unit length, $Q = \{q : ||q|| = 1\}$, act on $\mathbb{H}^0$ by conjugation $h \mapsto qhq^{-1}$. Show that this defines a rotation of $S^2 \subseteq \mathbb{H}^0$, so that $G/\{\pm I_2\} = Q/\{\pm I_2\} \cong \mathrm{SO}(3)$.

(b) [Needs some topological knowledge.]* First project S^2 onto its equatorial plane by $(x, y, z) \mapsto \zeta = \frac{x+iy}{1-z}$. Show that a rotation of S^2 corresponds to a transformation of the form $\zeta \mapsto \frac{a\zeta+b}{-\bar{b}\zeta+\bar{a}}$. Note that with $a\bar{a} + b\bar{b} = 1$ we obtain an element of G and that (a, b) and (a', b') determine the same transformation if and only if $(a', b') = (-a, -b)$. Now replace $G \cong S^3$ by the quotient space $\mathbb{R}P^3$.

7 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring. Find a generator for this polynomial ring.

8 It is known that any finite subgroup of SO(3) is isomorphic to precisely one of the following groups:

- the cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geqslant 1$, generated by a rotation by $2\pi/n$ around an axis;
- the dihedral group D_{2m} of order $2m$, $m \geqslant 2$ (the group of rotational symmetries in 3-space of a plane containing a regular m-gon);
- A_4, the group of rotations of a regular tetrahedron;
- S_4, the group of rotations of a cube (or regular octahedron);
- A_5, the group of rotations of a regular dodecahedron (or regular icosahedron).

Derive this classification (Hint: let G be a finite subgroup of SO(3) and consider the action of G on the unit sphere.) By considering the homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, classify the finite subgroups of SU(2).

9 The *Heisenberg group* of order p^3 is the (non-abelian) subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of matrices over the finite field $\mathbb{F}_p$ (p prime). Let H be the subgroup of G comprising matrices with $a = 0$ and Z be the subgroup of G of matrices with $a = b = 0$.

(a) Show that $Z = Z(G)$, the centre of G, and that $G/Z = \mathbb{F}_p^2$. Note that this implies that the derived subgroup G' is contained in Z. [You can check by explicit computation that it equals Z, or you can deduce this from the list of irreducible representations found in (d) below.]

(b) Find all 1-dimensional representations of G.

(c) Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ_ψ of H by

$$\rho_\psi \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $\mathrm{Ind}_H^G \rho_\psi$ is an irreducible representation of G.

(d) Prove that the collection of representations constructed in (b) and (c) gives a complete list of all irreducible representations.

(e) Determine the character of the irreducible representation $\mathrm{Ind}_H^G \rho_\psi$.

10 Recall the character table of $G = \mathrm{PSL}_2(7)$ from Sheet 2, q.9. Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $(\mathbb{F}_7)^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

*(Harder) Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.

[Hint: for the last part use the formula that the number of pairs of elements conjugate to x and y respectively, whose product is conjugate to t, equals $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, where the sum runs over all the irreducible characters of G, and $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]

11 Let $J_{\lambda,n}$ be the $n \times n$ Jordan block with eigenvalue $\lambda \in K$ (K is any field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

(a) Compute $J_{\lambda,n}^r$ for each $r \geqslant 0$.

(b) Let G be cyclic of order N, and let K be an algebraically closed field of characteristic $p > 0$. Determine *all* the representations of G on vector spaces over K, up to equivalence. Which are irreducible? Which are indecomposable?

Remark: Over $\mathbb{C}$ irreducibility and indecomposability coincide but this can fail for modular representations.

12 [For enthusiasts only. Part (a) requires knowledge of Galois Theory.]*

(a) Let G be a cyclic group and let χ be a (possibly reducible) character of G. Let $S = \{g \in G : G = \langle g \rangle\}$ and assume that $\chi(s) \neq 0$ for all $s \in S$. Show that

$$\sum_{s \in S} |\chi(s)|^2 \geqslant |S|.$$

(b) Deduce a theorem of Burnside: namely, let χ be an irreducible character of G with $\chi(1) > 1$. Show that $\chi(g) = 0$ for some $g \in G$. [Hint: partition G into equivalence classes by calling two elements of G equivalent if they generate the same cyclic subgroup of G.]

SM, Lent Term 2019
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`