

## PART II REPRESENTATION THEORY SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over  $\mathbb{C}$ . In the first seven questions we let  $G = \mathrm{SU}(2)$ . Questions 9 onwards deal with a variety of topics at Tripos standard.

**1** Let  $V_n$  be the vector space of complex homogeneous polynomials of degree  $n$  in the variables  $x$  and  $y$ . Describe a representation  $\rho_n$  of  $G$  on  $V_n$  and show that it is irreducible. What is its character? Show that  $V_n$  is isomorphic to its dual  $V_n^*$ .

**2** Decompose the representation  $V_4 \otimes V_3$  into irreducible  $G$ -spaces (that is, find a direct sum of irreducible representations which is isomorphic to  $V_4 \otimes V_3$ ; in this and the following questions, you are not being asked to find such an isomorphism explicitly). Decompose  $V_1^{\otimes n}$  into irreducibles.

**3** Determine the character of  $S^n V_1$  for  $n \geq 1$ .

Decompose  $S^2 V_n$  and  $\Lambda^2 V_n$  into irreducibles for  $n \geq 1$ .

Decompose  $S^3 V_2$  into irreducibles.

**4** Let  $G$  act on the space  $M_3(\mathbb{C})$  of  $3 \times 3$  complex matrices, by conjugation:

$$A : X \mapsto A_1 X A_1^{-1},$$

where  $A_1$  is the  $3 \times 3$  block diagonal matrix with block diagonal entries  $A, 1$ . Show that this gives a representation of  $G$  and decompose it into irreducibles.

**5** Let  $\chi_n$  be the character of the irreducible representation  $\rho_n$  of  $G$  on  $V_n$  of dimension  $n+1$ .

Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where  $z = e^{i\theta}$  and  $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$ .

[ Note that all you need to know about integrating on the circle is orthogonality of characters:  $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$ . This is really a question about Laurent polynomials. ]

**6** Check that the usual formula for integrating functions defined on  $S^3 \subseteq \mathbf{R}^4$  defines a  $G$ -invariant inner product on the vector space of integrable functions on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

**7** Compute the character of the representation  $S^n V_2$  of  $G$  for any  $n \geq 0$ . Calculate  $\dim_{\mathbb{C}} (S^n V_2)^G$  (by which we mean the subspace of  $S^n V_2$  where  $G$  acts trivially).

Deduce that the ring of complex polynomials in three variables  $x, y, z$  which are invariant under the action of  $\mathrm{SO}(3)$  is a polynomial ring. Find a generator for this polynomial ring.

**8** (a) Let  $G$  be a compact group. Show that there is a continuous group homomorphism  $\rho : G \rightarrow \mathrm{O}(n)$  if and only if  $G$  has an  $n$ -dimensional representation over  $\mathbb{R}$ . Here  $\mathrm{O}(n)$  denotes the subgroup of  $\mathrm{GL}_n(\mathbb{R})$  preserving the standard (positive definite) symmetric bilinear form.  
 (b) Explicitly construct such a representation  $\rho : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$  by showing that  $\mathrm{SU}(2)$  acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to  $\mathbb{R}^3$ , that  $(A, B) \mapsto -\mathrm{tr}(AB)$  is an invariant positive definite symmetric bilinear form, and that  $\rho$  is surjective with kernel  $\{\pm I\}$ .

**9** The *Heisenberg group* of order  $p^3$  is the (non-abelian) subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of matrices over the finite field  $\mathbb{F}_p$  ( $p$  prime). Let  $H$  be the subgroup of  $G$  comprising matrices with  $a = 0$  and  $Z$  be the subgroup of  $G$  of matrices with  $a = b = 0$ .

(a) Show that  $Z = Z(G)$ , the centre of  $G$ , and that  $G/Z = \mathbb{F}_p^2$ . Note that this implies that the derived subgroup  $G'$  is contained in  $Z$ . [You can check by explicit computation that it equals  $Z$ , or you can deduce this from the list of irreducible representations found in (d) below.]

(b) Find all 1-dimensional representations of  $G$ .

(c) Let  $\psi : \mathbb{F}_p \rightarrow \mathbb{C}^\times$  be a non-trivial 1-dimensional representation of the cyclic group  $\mathbb{F}_p = \mathbb{Z}/p$ , and define a 1-dimensional representation  $\rho_\psi$  of  $H$  by

$$\rho_\psi \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that  $\mathrm{Ind}_H^G \rho_\psi$  is an irreducible representation of  $G$ .

(d) Prove that the collection of representations constructed in (b) and (c) gives a complete list of all irreducible representations.

(e) Determine the character of the irreducible representation  $\mathrm{Ind}_H^G \rho_\psi$ .

**10** Recall the character table of  $G = \mathrm{PSL}_2(7)$  from Sheet 2, q.8. Identify the columns corresponding to the elements  $x$  and  $y$  where  $x$  is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and  $y$  is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group  $G$  acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space  $(\mathbb{F}_7)^2$ ). Obtain the permutation character of this action and decompose it into irreducible characters.

\*(Harder) Show that the group  $G$  is generated by an element of order 2 and an element of order 3 whose product has order 7.

[Hint: for the last part use the formula that the number of pairs of elements conjugate to  $x$  and  $y$  respectively, whose product is conjugate to  $t$ , equals  $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$ , where the sum runs over all the irreducible characters of  $G$ , and  $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$ .]

11 Let  $J_{\lambda,n}$  be the  $n \times n$  Jordan block with eigenvalue  $\lambda \in K$  ( $K$  is any field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

(a) Compute  $J_{\lambda,n}^r$  for each  $r \geq 0$ .

(b) Let  $G$  be cyclic of order  $N$ , and let  $K$  be an algebraically closed field of characteristic  $p > 0$ . Determine *all* the representations of  $G$  on vector spaces over  $K$ , up to equivalence. Which are irreducible? Which are indecomposable?

Remark: Over  $\mathbb{C}$  irreducibility and indecomposability coincide but this can fail for modular representations.

12 [For enthusiasts only. Part (a) requires knowledge of Galois Theory.]

(a) Let  $G$  be a cyclic group and let  $\chi$  be a (possibly reducible) character of  $G$ . Let  $S = \{g \in G : G = \langle g \rangle\}$  and assume that  $\chi(s) \neq 0$  for all  $s \in S$ . Show that

$$\sum_{s \in S} |\chi(s)|^2 \geq |S|.$$

(b) Deduce a theorem of Burnside: namely, let  $\chi$  be an irreducible character of  $G$  with  $\chi(1) > 1$ . Show that  $\chi(g) = 0$  for some  $g \in G$ . [Hint: partition  $G$  into equivalence classes by calling two elements of  $G$  equivalent if they generate the same cyclic subgroup of  $G$ .]

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