PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first seven questions we let $G = \mathrm{SU}(2)$.

1 Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. What is its character? Show that V_n is isomorphic to its dual V_n^*.

2 Decompose the representation $V_4 \otimes V_3$ into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$; in this and the following questions, you are not being asked to find such an isomorphism explicitly). Decompose $V_1^{\otimes n}$ into irreducibles.

3 Determine the character of $S^n V_1$ for $n \geqslant 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ into irreducibles for $n \geqslant 1$.
Decompose $S^3 V_2$ into irreducibles.

4 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by conjugation:

$$A : \ X \ \mapsto \ A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

5 Let χ_n be the character of the irreducible representation ρ_n of G on V_n of dimension $n+1$.

Show that

$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi}\int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

6 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines a G-invariant inner product on the vector space of integrable functions on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

7 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of $\mathrm{SO}(3)$ is a polynomial ring. Find a generator for this polynomial ring.

8 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive definite) symmetric bilinear form.
(b) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbb{R}^3$, that $(A, B) \mapsto -\mathrm{tr}(AB)$ is an invariant positive definite symmetric bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

9 The *Heisenberg group* of order p^3 is the (non-abelian) subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of matrices over the finite field $\mathbb{F}_p$ (p prime). Let H be the subgroup of G comprising matrices with $a = 0$ and Z be the subgroup of G of matrices with $a = b = 0$.

(a) Show that $Z = Z(G)$, the centre of G, and that $G/Z = \mathbb{F}_p^2$. Note that this implies that the derived subgroup G' is contained in Z. [You can check by explicit computation that it equals Z, or you can deduce this from the list of irreducible representations found in (d) below.]

(b) Find all 1-dimensional representations of G.

(c) Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ_ψ of H by

$$\rho_\psi \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $\mathrm{Ind}_H^G \rho_\psi$ is an irreducible representation of G.

(d) Prove that the collection of representations constructed in (b) and (c) gives a complete list of all irreducible representations.

(e) Determine the character of the irreducible representation $\mathrm{Ind}_H^G \rho_\psi$.

10 Recall Sheet 3, q.8 where we used inner products to construct some irreducible characters $\chi^{(n-r,r)}$ for S_n. Let $n \in \mathbb{N}$, and let Ω be the set of all ordered pairs (i, j) with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let $G = S_n$ act on Ω in the obvious manner (namely, $\sigma(i, j) = (\sigma i, \sigma j)$ for $\sigma \in S_n$). Let's write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action.

Prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Writing $\psi = \chi^{(n-2,1,1)}$, calculate the degree of $\chi^{(n-2,1,1)}$. Find its value on any transposition and on any 3-cycle. Returning to the character table of S_6 calculated on Sheet 3, identify the character $\chi^{(4,1,1)}$.

11 Recall the character table of $G = \mathrm{PSL}_2(7)$ from Sheet 2, q.8. Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $(\mathbb{F}_7)^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

*(Harder) Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.
[Hint: for the last part use the formula that the number of pairs of elements conjugate to x and y respectively, whose product is conjugate to t, equals $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, where the sum runs over all the irreducible characters of G, and $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]

12 [For enthusiasts only.]

It is known that the classification of finite subgroups of SO(3) is as follows:

- the cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geqslant 1$, generated by a rotation by $2\pi/n$ around an axis;
- the dihedral group D_{2m} of order $2m$, $m \geqslant 2$ (the group of rotational symmetries in 3-space of a plane containing a regular m-gon);
- A_4, the group of rotations of a regular tetrahedron;
- S_4, the group of rotations of a cube or regular octahedron;
- A_5, the group of rotations of a regular dodecahedron or icosahedron.

By considering the homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, classify[1] the finite subgroups of SU(2).

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Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`

[1]Alternatively, as seen in Part III, a correspondence due to McKay gives a bijection between the finite subgroups of SU(2) and the so-called affine simply laced Dynkin diagrams. The bijection associates naturally to each finite-dimensional representation of SU(2) a vertex of the corresponding diagram.