PART II REPRESENTATION THEORY
SHEET 2

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d, and affording character χ. Show that $\ker \rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leqslant d$ for all $g \in G$, with equality only if $\rho(g) = \lambda I$, a scalar multiple of the identity, for some root of unity λ.

2 Let χ be the character of a representation V of G and let g be an element of G. If g is an involution (i.e. $g^2 = 1 \neq g$), show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: consider the determinant of g acting on V.) If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

3 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. You should notice something interesting.

4 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for $D_{12} \cong S_3 \times C_2$ and the cyclic subgroup C_6 of D_{12}.

5 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

6 A certain group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β. The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

7 The table below is a part of the character table of a certain finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

8 Let x be an element of order n in a finite group G. Say, without detailed proof, why

(a) if χ is a character of G, then $\chi(x)$ is a sum of nth roots of unity;

(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};

(c) x and x^{-1} have the same number of conjugates in G.

Prove that the number of irreducible characters of G which take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes (so-called *real classes*).

9 A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.
[You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of unity.]

The character table thus obtained is in fact the character table of the group $G = \mathrm{PSL}_2(7)$ of 2×2 matrices with determinant 1 over the field $\mathbb{F}_7$ (of seven elements) modulo the two scalar matrices. Deduce directly from the character table that G is simple[1].

10 The group M_9 is a certain subgroup of the symmetric group S_9 generated by the two elements $(1,4,9,8)(2,5,3,6)$ and $(1,6,5,2)(3,7,9,8)$. You are given the following facts about M_9:

• there are six conjugacy classes:

— C_1 contains the identity.

— For $2 \leqslant i \leqslant 4$, $|C_i| = 18$ and C_i contains g_i, where $g_2 = (2,3,8,6)(4,7,5,9)$, $g_3 = (2,4,8,5)(3,9,6,7)$ and $g_4 = (2,7,8,9)(3,4,6,5)$.

— $|C_5| = 9$, and C_5 contains $g_5 = (2,8)(3,6)(4,5)(7,9)$

— $|C_6| = 8$, and C_6 contains $g_6 = (1,2,8)(3,9,4)(5,7,6)$.

• every element of M_9 is conjugate to its inverse.

Calculate the character table of M_9. [Hint: You may find it helpful to notice that $g_2^2 = g_3^2 = g_4^2 = g_5$.]

11 Let a finite group G act on itself by conjugation. Find the character of the corresponding permutation representation.

12 Consider the character table Z of G as a matrix of complex numbers (as we did when deriving the column orthogonality relations from the row orthogonality relations).

(a) Using the fact that the complex conjugate of an irreducible character is also an irreducible character, show that the determinant $\det Z$ is $\pm \det \bar{Z}$, where $\bar{Z}$ is the complex conjugate of Z.

(b) Deduce that either $\det Z \in \mathbb{R}$ or $\det Z \in i\mathbb{R}$.

(c) Use the column orthogonality relations to calculate the product $\bar{Z}^T Z$, where $\bar{Z}^T$ is the transpose of the complex conjugate of Z.

(d) Calculate $|\det Z|$.

[1] It is known that there are precisely five non-abelian simple groups of order less than 1000. The smallest of these is $A_5 \cong \mathrm{PSL}_2(5)$, while G is the second smallest. It is also known that for $p \geqslant 5$, $\mathrm{PSL}_2(p)$ is simple.

13 The group $\mathrm{SL}_2(\mathbb{F}_q)$ acts on the projective line $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Show that $\mathrm{SL}_2(\mathbb{F}_q)$ has an irreducible representation of dimension q. [Hint: use the corollary to Burnside's Lemma.]

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