PART II REPRESENTATION THEORY
SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.

(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.

(b) Prove that $G/\ker\delta$ is abelian.

(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta\otimes\rho : G \to\mathrm{GL}(V)$ given by $\theta\otimes\rho : g \mapsto \theta(g)\cdot\rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 (Counterexamples to Maschke's Theorem)

(a) Let FG denote the regular FG-module (i.e. the permutation module coming from the action of G on itself by left multiplication), and let F be the trivial module. Find all the FG-homomorphisms from FG to F and vice versa. By considering a submodule of FG isomorphic to F, prove that whenever the characteristic of F divides the order of G, there is a counterexample to Maschke's Theorem.

(b) Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

4 Describe Weyl's unitary trick.

Let G be a finite group acting on a complex vector space V, and let $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ be a skew-symmetric form, i.e. $\langle\ y,\ x\ \rangle = -\langle\ x,\ y\ \rangle$ for all x, y in V.

Show that the form $(x, y) = \frac{1}{|G|}\sum\langle gx, gy\rangle$, where the sum is over all elements $g \in G$, is a G-invariant skew-symmetric form.

Does this imply that every finite subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ is conjugate to a subgroup of the symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$?

5 Let $G = \mathbb{Z}/n$ be a cyclic group of order n. Explicity decompose the (complex) regular representation of G as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

6 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y\ :\ x^5 = 1 = y^2,\ yxy^{-1} = x^{-1}\rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

7 Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2,\ yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

8 Show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

9 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

10 In this question work over the field $F = \mathbb{R}$.

Let G be the cyclic group of order 3.

(a) Write the regular $\mathbb{R}G$-module as a direct sum of irreducible submodules.

(b) Find all the $\mathbb{R}G$-homomorphisms between the irreducible $\mathbb{R}G$-modules.

(c) Show that the conclusion of Schur's Lemma ('every homomorphism from an irreducible module to itself is a scalar multiple of the identity') is false if you replace $\mathbb{C}$ by $\mathbb{R}$.

From now on let G be a cyclic group of order n. Show that:

(d) If n is even, the regular $\mathbb{R}G$-module is a direct sum of two (non-isomorphic) 1-dimensional irreducible submodules and $(n-2)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

(e) If n is odd, the regular $\mathbb{R}G$-module is a direct sum of one 1-dimensional irreducible submodule and $(n-1)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

[Hint: use the fact that $\mathbb{R}G \subset \mathbb{C}G$ and what you know about the regular $\mathbb{C}G$-module from question 5.]

11 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

12 Let $J_{\lambda,n}$ be the $n \times n$ Jordan block with eigenvalue $\lambda \in K$ (K is any field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

(a) Compute $J^r_{\lambda,n}$ for each $r \geqslant 0$.

(b) Let $G = \mathbb{Z}/N$ be cyclic of order N, and let K be an algebraically closed field of characteristic $p \geqslant 0$. Determine all the representations of G on vector spaces over K, up to equivalence. Which are irreducible?

13 A hermitian inner product on $\mathbb{C}^2$ is given by a 2×2 matrix X such that $\bar{X}^T = X$; the inner product is $\langle x, y\rangle = x^T X \bar{y}$. Explicitly find a hermitian inner product invariant under the group $G \leqslant \mathrm{GL}_2(\mathbb{C})$ generated by the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

[Hint: average the standard hermitian inner product.]

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Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`