PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first nine questions we let $G = \mathrm{SU}(2)$.

1 (a) Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. Describe the character χ_n of ρ_n.

(b) Decompose $V_4 \otimes V_3$ into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$. In this and the following questions, you are not being asked to find such an isomorphism explicitly.)

(c) Decompose also $V_3^{\otimes 2}$, $\Lambda^2 V_3$ and $S^2 V_3$.

(d) Show that V_n is isomorphic to its dual V_n^*.

2 Decompose $V_1^{\otimes n}$ into irreducibles.

3 Determine the character of $S^n V_1$ for $n \geq 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ for $n \geq 1$.
Decompose $S^3 V_2$ into irreducibles.

4 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by

$$A : \ X \ \mapsto \ A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

5 Let χ_n be the character of the irreducible representation ρ_n of G on V_n.
Show that

$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\chi_n\overline{\chi_m}d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi}\int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

6 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive definite) symmetric bilinear form.
(b) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbb{R}^3$, that $(A, B) \mapsto -\mathrm{tr}(AB)$ is a positive definite non-degenerate invariant bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

7 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines an G-invariant inner product on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

8 Suppose we are given that H is a subgroup of order 24 in G. We are told that H contains $\{\pm I\}$ as a normal subgroup, and that the quotient group $H/\{\pm I\}$ is isomorphic to A_4. Find the character table of H. [You may assume that H has a conjugacy class containing six elements of order 4, two conjugacy classes each containing four elements of order 3, and two conjugacy classes each containing four elements of order 6.]

9 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring. Find a generator for this polynomial ring.

10 The *Heisenberg group* of order p^3 is the (non-abelian) group

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of 3×3 upper unitriangular matrices over the finite field $\mathbb{F}_p$ of p elements (p prime).

Show that G has p conjugacy classes of size 1, and $p^2 - 1$ conjugacy classes of size p.

Find p^2 characters of degree 1.

Let H be the subgroup of G comprising matrices with $a = 0$. Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Check that $V_\psi = \mathrm{Ind}_H^G \rho$ is irreducible.

Now list all the irreducible representations of G, explaining why your list is complete.

11 Recall Sheet 3, q.8 where we used inner products to construct some irreducible characters $\chi^{(n-r,r)}$ for S_n. Let $n \in \mathbb{N}$, and let Ω be the set of all ordered pairs (i, j) with $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Let $G = S_n$ act on Ω in the obvious manner (namely, $\sigma(i, j) = (\sigma i, \sigma j)$ for $\sigma \in S_n$). Let's write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action.

Prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Writing $\psi = \chi^{(n-2,1,1)}$, calculate the degree of $\chi^{(n-2,1,1)}$. Find its value on any transposition and on any 3-cycle. Returning to the character table of S_6 calculated on Sheet 3 q.11, identify the character $\chi^{(4,1,1)}$.

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Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`