PART II REPRESENTATION THEORY
SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.

(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.

(b) Prove that $G/\ker\delta$ is abelian.

(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta\otimes\rho : G \to \mathrm{GL}(V)$ given by $\theta\otimes\rho : g \mapsto \theta(g)\cdot\rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 Let G be the alternating group A_4. Find all the degree one representations of G over F for:

(a) $F = \mathbb{C}$;

(b) $F = \mathbb{R}$;

(c) $F = \mathbb{Z}/3\mathbb{Z}$.

[Hint: you can use the fact that the Klein 4-group $V = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is the unique normal subgroup of A_4 (apart from the trivial subgroup and A_4 itself).]

Now let $G = D_{12}$, the symmetry group of a regular hexagon. Let $a \in G$ be a rotation through $\pi/3$ anticlockwise, and let $b \in G$ be a reflection, so that $G = \{a^i, a^ib \ : \ 0 \leqslant i \leqslant 5\}$. Let $A, B, C, D \in \mathrm{GL}_2(\mathbb{C})$ be the matrices

$$A = \begin{pmatrix} e^{\pi i/3} & 0 \\ 0 & e^{-\pi i/3} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each of the following is a (matrix) representation of G over $\mathbb{C}$ (you need not verify this):

$\rho_1 : a^rb^s \mapsto A^rB^s$.

$\rho_2 : a^rb^s \mapsto A^{3r}(-B)^s$.

$\rho_3 : a^rb^s \mapsto (-A)^rB^s$.

$\rho_4 : a^rb^s \mapsto C^rD^s$.

Which of these are faithful? Which are equivalent to one another?

4 (Counterexamples to Maschke's Theorem)

(a) Let FG denote the regular FG-module (i.e. the permutation module coming from the action of G on itself by left multiplication), and let F be the trivial module. Find all the FG-homomorphisms from FG to F and vice versa. By considering a submodule of FG isomorphic to F, prove that whenever the characteristic of F divides the order of G, there is a counterexample to Maschke's Theorem.

(b) Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

5 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G'.

6 Let G be a cyclic group of order n. Decompose the regular representation of G explicitly as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g\in G}$ to a basis where the group action is diagonal.

7 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y \ : \ x^5 = 1 = y^2, \ yxy^{-1} = x^{-1}\rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

8 Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, \ yxy^{-1} = x^{-1}\rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

9 State Schur's lemma.

Show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

10 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56)\rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

11 In this question work over the field $F = \mathbb{R}$.

Let G be the cyclic group of order 3.

(a) Write the regular $\mathbb{R}G$-module as a direct sum of irreducible submodules.

(b) Find all the $\mathbb{R}G$-homomorphisms between the irreducible $\mathbb{R}G$-modules.

(c) Show that the conclusion of Schur's Lemma ('every homomorphism from an irreducible module to itself is a scalar multiple of the identity') is false if you replace $\mathbb{C}$ by $\mathbb{R}$.

From now on let G be a cyclic group of order n. Show that:

(d) If n is even, the regular $\mathbb{R}G$-module is a direct sum of two (non-isomorphic) 1-dimensional irreducible submodules and $(n-2)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

(e) If n is odd, the regular $\mathbb{R}G$-module is a direct sum of one 1-dimensional irreducible submodule and $(n-1)/2$ (non-isomorphic) 2-dimensional irreducible submodules.

[Hint: use the fact that $\mathbb{R}G \subset \mathbb{C}G$ and what you know about the regular $\mathbb{C}G$-module from question 6.]

12 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^tA = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

SM, Lent Term 2011
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