PART II REPRESENTATION THEORY
SHEET 2

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d, and affording character χ. Show that $\ker\rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leqslant d$ for all $g \in G$, with equality only if $\rho(g) = \lambda I$, a scalar multiple of the identity, for some root of unity λ.

2 Let χ be the character of a representation V of G and let g be an element of G. If g has order 2, show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: consider the determinant of g acting on V.) If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

3 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. You should notice something interesting.

4 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for $D_{12} \cong S_3 \times C_2$ and the cyclic subgroup C_6 of D_{12}.

5 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

6 A certain group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β. The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

7 The table below is a part of the character table of a certain finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

8 Let x be an element of order n in a finite group G. Say, without detailed proof, why
(a) if χ is a character of G, then $\chi(x)$ is a sum of nth roots of unity;
(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};
(c) x and x^{-1} have the same number of conjugates in G.

Prove that the number of irreducible characters of G which take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes (so-called *real classes*).

A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.
[You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of unity.]

9 Let a finite group G act on itself by conjugation. Find the character of the corresponding permutation representation.

10 Let G have conjugacy class representatives $g_1, \ldots, g_k$ and character table Z. Show that $\det Z$ is either real or purely imaginary, and that

$$|\det Z|^2 = \prod_{i=1}^{k} |C_G(g_i)|.$$

11 The character table obtained in Question 8 is in fact the character table of the group $G = \mathrm{PSL}_2(7)$ of 2×2 matrices with determinant 1 over the field $\mathbb{F}_7$ (of seven elements) modulo the two scalar matrices.

Deduce directly from the character table which you have obtained that G is simple.

[Comment: it is known that there are precisely five non-abelian simple groups of order less than 1000. The smallest of these is $A_5 \cong \mathrm{PSL}_2(5)$, while G is the second smallest. It is also known that for $p \geqslant 5$, $\mathrm{PSL}_2(p)$ is simple.]

Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $(\mathbb{F}_7)^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.

[Hint: for the last part use the formula that the number of pairs of elements conjugate to x and y respectively, whose product is conjugate to t, equals $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, where the sum runs over all the irreducible characters of G, and $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]

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Comments on and corrections to this sheet may be emailed to sm@dpmms.cam.ac.uk