Part II Representation Theory Sheet 4

Unless otherwise stated, all vector spaces are finite dimensional over $\mathbf{C}$.

Q.1 Let $G = SU(2)$, let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. Describe the character χ_n of ρ_n.

Q.2 Decompose $V_4 \otimes V_3$ into irreducible G-spaces. (That is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$. In this and the following questions, you are not being asked to find such an isomorphism explicitly.)
Decompose $V_3^{\otimes 2}$, $\Lambda^2 V_3$ and $S^2 V_3$.

Q.3 Decompose $V_1^{\otimes n}$ into irreducibles.

Q.4 Determine the character of $S^n V_1$ for $n \geq 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ for $n \geq 1$.
Decompose $S^3 V_2$ into irreducibles.

Q.5 Let $G = SU(2)$ act on the space $M_3(\mathbf{C})$ of 3×3 complex matrices, by

$$A : \; X \;\mapsto\; A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

Q.6 Let $G = SU(2)$. Show that V_n is isomorphic to its dual V_n^*.

Q.7 Let $G = SU(2)$, and let χ_n be the character of the irreducible representation ρ_n of G on V_n.
Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

Q.8 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to O(n)$ if and only if G has an n-dimensional representation over $\mathbf{R}$. Here $O(n)$ denotes the subgroup of $GL(n, \mathbf{R})$ preserving the standard (positive definite) symmetric bilinear form.

(b) Explicitly construct such a representation $\rho : SU(2) \to SO(3)$ by showing that $SU(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbf{C}) \mid A + \overline{A}^t = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbf{R}^3$, that $(A, B) \mapsto -tr(AB)$ is a positive definite non-degenerate invariant bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

Q.9 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines an $SU(2)$-invariant inner product on

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

Q.10 The *Heisenberg group* is the group G of order p^3 of upper unitriangular matrices over the field of p elements. Show that G has p conjugacy classes of size 1, and $p^2 - 1$ conjugacy classes of size p.
Find p^2 characters of degree 1. Show that there are $p - 1$ irreducible characters of G of degree p induced from 1-dimensional characters of the abelian subgroup of matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.