Part II Representation Theory Sheet 2

Unless otherwise stated, all vector spaces are finite dimensional over a field F of characteristic zero, usually $\mathbf{C}$.

Q.1 Let $\rho : G \to GL(V)$ be a representation of G of dimension d, with character χ. Show that $\ker\rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leq d$ for all $g \in G$, with equality only if $\rho(g) = \lambda I$, a scalar multiple of the identity, for some root of unity λ.

Q.2 Let χ be the character of a representation V of G and let g be an element of G. If g has order 2, show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: Consider the determinant of g acting on V.) If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

Q.3 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. Comment.

Q.4 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for D_{12} and the cyclic subgroup C_6 of D_{12}.

Q.5 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

Q.6 A group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β. The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

Q.7 The table below is a part of the character table of a finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

Q.8 Let x be an element of order n in a finite group G. Say, without detailed proof, why

(a) if χ is a character of G, then $\chi(x)$ is a sum of n-th roots of unity;

(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};

(c) x and x^{-1} have the same number of conjugates in G.

State the orthogonality relations that hold between the rows and columns of the character table of G.

A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.

[*You may assume, if needed, the fact that* $\sqrt{7}$ *is not in the field* $\mathbf{Q}(\zeta)$, *where* ζ *is a primitive 7th root of unity.*]

Q.9 Let a finite group G act on itself by conjugation and find the character of the corresponding permutation representation. Prove that the sum of the elements in any row of the character table for G is a non-negative integer.

Q.10 Show that the complex character table of a finite group G is invertible when viewed as a matrix.

Prove that the number of irreducible characters of G which take only real values is equal to the number of self-inverse conjugacy classes.

[*Consider the permutation action induced by complex conjugation on rows and on columns.*]

Q.11 The character table obtained in Question 8 is the character table of the group $G = PSL_2(7)$ of 2×2 matrices with determinant 1 over the field $\mathbf{F}_7$ of seven elements modulo the scalars.

Use the character table which you have obtained to show that this group is simple.

Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $\mathbf{F}_7^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.

[*For the last part use the formula that the number of pairs of elements conjugate to* x *and* y *respectively, whose product is conjugate to* t, *equals* $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, *where the sum runs over all the irreducible characters of* G, *and* $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]