Representation Theory Sheet 2

Unless otherwise stated, all vector spaces are finite dimensional over a field F of characteristic zero, usually $\mathbf{C}$.

Q.1 Let $\rho : G \to GL(V)$ be a representation of G of dimension d, with character χ. Show that $\ker\rho = \{g \in G \mid \chi(g) = d\}$. Show further that $|\chi(g)| \leq d$ for all $g \in G$, with equality only if $\rho(g) = \lambda I$, a scalar multiple of identity, for some root of unity λ.

Q.2 Let χ be a character of G, let $g \in G$. If g has order 2, show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$; if G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

Q.3 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. Comment.

Q.4 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for D_{12} and the cyclic subgroup C_6 of D_{12}.

Q.5 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

Q.6 A group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β, The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

Q.7 The table below is a part of the character table of a finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0
χ_3	3	$\bar{\gamma}$	γ	0	0

Q.8 Let x be an element of order n in a finite group G. Say, without detailed proof, why

(a) if χ is a character of G, then $\chi(x)$ is a sum of n-th roots of unity;

(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};

(c) x and x^{-1} have the same number of conjugates in G.

State the orthogonality relations that hold between the rows and columns of the character table of G.

A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.

[*You may assume, if needed, the fact that* $\sqrt{7}$ *is not in the field* $\mathbf{Q}(\zeta)$*, where* ζ *is a primitive 7th root of unity.*]

Q.9 Let a finite group G act on itself by conjugation and find the character of the corresponding permutation representation. Prove that the sum of the elements in any row of the character table for G is a non-negative integer.

Q.10 Show that the complex character table of a finite group G is invertible when viewed as a matrix.

Prove that the number of irreducible characters of G which take only real values is equal to the number of self-inverse conjugacy classes.

[*Consider the permutation action induced by complex conjugation on rows and on columns.*]

Q.11 The character table obtained in Question 8 is the character table of the group $G = PSL_2(7)$ of non-singular 2×2 matrices over the field F_7 of seven elements modulo the scalars.

Give two proofs based on the character table which you have obtained that this group is simple.

Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space F_7^2). Obtain the permutation character of this action and decompose it into irreducible characters.

Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.

[*For the last part use the formula that the number of pairs of elements conjugate to* x *and* y *respectively, whose product is conjugate to* t*, equals* $c\sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$*, where the sum runs over all the irreducible characters of* G*, and* $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]