Representation Theory Sheet 4

4.1 Question

Show that any continuous group homomorphisms $\phi : U(1) \to \mathbb{C}^\times$ lands in the subgroup $U(1)$ of unit norm complex numbers. Moreover, show that: either $\phi \equiv 1$, or else $\ker \phi$ is a cyclic subgroup of roots of unity.

4.2 Question

Let $G = SU_2$ and let V_n be its irreducible $(n+1)$-dimensional representation. (So $V_0 = 1$). Decompose the following representations into irreducibles:

(a) $V_2 \otimes V_4$ b) $V_3 \otimes V_4$ (c) $\mathrm{Sym}^2 V_3$ (d) $\Lambda^2 V_3$ (e) $V_1^{\otimes n}$ (f) $\mathrm{Sym}^n V_1$ (g) $\mathrm{Sym}^3 V_2$.

4.3 Question

Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines an SU_2 invariant inner product on $SU_2 = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a\bar{a} + b\bar{b} = 1\}$, and normalize it so that the integral over the group is one.

4.4 Question

Let $G = SU_2$, and χ_n be the character of the irreducible representation V_n. Show that

$$\int_{S^1} K(z)\chi_n\overline{\chi_m}dz = \delta_{nm},$$

where $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\int_{S^1} z^n dz = \delta_{n,0}$. *This is really a question about Laurent polynomials!*]

4.5 Question

Let $G = SU_2$. Show that $V_n \simeq V_n^*$.

4.6 Question

(a) Let G be a compact group. Show that there is a group homomorphism $\rho : G \to O_n(\mathbf{R})$ if and only if G has an n dimensional representation over $\mathbf{R}$.
(b) Explicitly construct such a representation $\rho : SU_2 \to SO_3(\mathbf{R})$ by showing that SU_2 act on the vector space of matrices of the form

$$\{A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{C}) \mid A + \overline{A}^T = 0\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbf{R}^3$, that $(A, B) \mapsto -tr(AB)$ is a positive definite non-degenerate invariant bilinear form, and that ρ is surjective with kernel $\mathbf{Z}/2$.

4.7 Question

Let $\mathbf{H}$ denote the quaternions. Show that the map

$$SU_2 \to \mathbf{H}, \qquad \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + bj$$

defines a group homomorphism, which identifies SU_2 with the quaternions of unit norm.

4.8 Question

Let $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in F_p^*, b \in F_p\}$, with multiplication given by matrix multiplication.
i) Determine the conjugacy classes in G.
ii) G has F_p as a normal subgroup N, and F_p^* is its quotient. Use this to produce $p-1$ one dimensional representations of G.
iii) Let $\phi : F_p \to \mathbf{C}^*$ be a non-trivial group homomorphism, $\phi(x+y) = \phi(x)\phi(y)$. This defines a one dimensional representation of N. Decompose $\mathrm{Ind}_N^G \phi$ into irreducibles. How does this depend on ϕ?
iv) Determine the character table of G.

4.9 Question

The *Heisenberg group* is the group

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \mid a, b, c \in F_p \right\}.$$

(a) Determine the conjugacy classes in H.
(b) Determine the center of H, and the quotient H/Z_H of H by its center.
(c) Let M be the subset of H given by matrices such that $b = 0$, and let $\phi : F_p \to \mathbf{C}^*$ be a non-trivial group homomorphism. Show that M is a subgroup, and that the map $M \to \mathbf{C}^*$, $\begin{pmatrix} 1 & a & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto \phi(c)$ is a group homomorphism, and hence determines a one dimensional representation $\tilde{\phi}$ of M.

Determine the character of $\mathrm{Ind}_M^H \tilde{\phi}$, and show it is an irreducible representation.
(d) Write the character table for H.
Hint: You will find p^2 1-dimensional representations, and $(p-1)$ of dimension p.

4.10 Question

Let B be the invertible upper triangular n by n matrices over F_p, and U the upper triangular matrices with 1's on the diagonal.
i) Show that U is nilpotent, and determine its central series.
ii) Show that B is solvable, and determine its derived series.