Representation Theory Sheet 3

G is a finite group and vector spaces are finite-dimensional over $\mathbb{C}$.

3.1 Question

Let $G = A_5$. Let χ be the character of one of the 3 dimensional irreducible representations, χ' the character of the 4 dimensional irreducible representation.
Decompose $\chi \otimes \chi$, $\chi \otimes \chi'$, and $\chi' \otimes \chi'$ into irreducible representations.
Decompose $\bigwedge^2 \chi$, $S^2\chi$ into irreducible represntations.
You will need to know the character table of A_5 to do this question!

3.2 Question

(i) For each irreducible representation ρ of A_4, determine the character of $\mathrm{Ind}_{A_4}^{S_4}\rho$ and decompose this into irreducible representations.
(ii) Now do this for $A_5 \leq S_5$, and $S_3 \leq S_4$.
(iii) For each irreducible representation $\tilde{\rho}$ of S_4, decompose $\mathrm{Res}_{A_4}^{S_4}\tilde{\rho}$ into irreducible representations. Check your answer is compatible with Frobenius reciprocity.

3.3 Question

Determine the character table of the dihedral group D_{2n} of symmetries of the n-gon. For each irreducible representation of $\mathbf{Z}/n$, decompose the induced representation of D_{2n}. *Note: It matters whether n is even or odd.*

3.4 Question

Show that the subgroup of A_5 generated by (12345) and $(25)(34)$ is isomorphic to D_{10}, and describe the induction of irreducible representations from this D_{10} to A_5.

3.5 Question

Let V be a representation of a group G.
(i) Compute $\dim S^nV$, $\dim \bigwedge^n V$.
(ii) Let $g \in G$. Suppose g has eigenvalues $\lambda_1, \dots, \lambda_d$ on V. What are the eigenvalues of g on S^nV, $\bigwedge^n V$?
(iii) Suppose $f(x) = \det(g - xI)$ is the characteristic polynomial of g. Describe how to read $tr(g, \bigwedge^n V)$ from the coefficients of $f(x)$.
(iv) Find a relation between $tr(g, S^nV)$ and the polynomial $f(x)$. (First do the case $\dim V = 1$.)

3.6 Question

Prove that there is a natural isomorphism (not requiring any choices) $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

3.7 Question

Let V be a finite dimensional representation of G and let W_k, $k = 1, \dots, r$ be a complete set of irreducible reps up to isomorphism. Define a natural morphism of G-representations

$$\bigoplus_k \mathrm{Hom}_G(W_k, V) \otimes W_k \to V$$

and show it is an isomorphism. This shows $\mathrm{Hom}_G(W_k, V) \otimes W_k$ is naturally isomorphic to the W_k-isotypical component of V.

3.8 Question

Let ρ and σ be representations of two finite groups G and H on complex vector spaces V and W. Define a representation $\rho \otimes \sigma$ of the product group $G \times H$ on $V \otimes W$ by $(\rho \otimes \sigma)(g, h) := \rho(g) \otimes \sigma(h)$. Determine the character of $\rho \otimes \sigma$ and, using this, show that it is irreducible if ρ and σ are so. Show that every irreducible representation of $G \times H$ is of this form.
How do you reconcile this with the example in class that the tensor square G-representation $W \otimes W$ can be *reducible*, even if W was irreducible?

3.9 Question

The group $G \times G$ acts on G by $(g, h) \cdot x = gxh^{-1}$, and so the regular representation $\mathbb{C}[G]$ is a representation of $G \times G$. *Until now, we've only considered* $\mathbb{C}[G]$ *as a representation of the group* $G \times \{1\} \leq G \times G$.
i) Determine its character.
ii) For each irreducible representation $V \otimes W$ of $G \times G$, determine the inner product of its character with that of $\mathbb{C}[G]$.
iii) Hence decompose $\mathbb{C}[G]$ as a representation of $G \times G$.

3.10 Question

Prove that the decomposition of $\mathbb{C}[G]$ you've just found is canonical, by showing that the map

$$\mathbb{C}[G] \to \bigoplus_V \operatorname{End}(V)$$

defined by sending e_g to the element whose V'th entry is $\rho_V(g)$ is a $G \times G$-map which has as inverse the map which sends $\phi \in \operatorname{End}(V)$ to the element $\sum_g \frac{1}{\dim V} \operatorname{Tr}_V(\phi \rho_V(g^{-1})) e_g$.
Notice that this explains why every irreducible representation V *of* $G \times \{1\}$ *occurs* $\dim V$ *times.*

3.11 Question

Prove the transitivity of induction: if $K \subset H \subset G$ are subgroups, then $\operatorname{Ind}_H^G \operatorname{Ind}_K^H = \operatorname{Ind}_K^G$.

3.12 Question

i) Carefully prove the Mackey formula for $\operatorname{Res}_K^G \operatorname{Ind}_H^G V$ by completing the proof sketched in class.
ii) Give another proof by computing characters.