Representation Theory, Sheet 2, 2006

G is a finite group and vector spaces are finite-dimensional over $\mathbb{C}$.

2.1 Question

Let $\rho : G \to GL(V)$ be an irreducible representation of G with character χ, and let $d = \dim V$.
(a) Show that $\ker \rho = \{g \in G \mid \chi(g) = d\}$.
(b) Show that $|\chi(g)| \leq d$ for all $g \in G$, and that if $|\chi(g)| = d$ then $\rho(g) = \lambda I$, where λ is a root of unity.

2.2 Question

Let χ be a character of G, and suppose $g \in G$ has order 2, ie. that $g^2 = 1$. Show that $\chi(g) \in \mathbf{Z}$, and that $\chi(g) \equiv \chi(1) \mod 2$.

2.3 Question

(a) Let G be the "quaternion group",

$$Q_8 := \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\},$$

where $ij = k = -ji, i^2 = j^2 = k^2 = -1$. Regarding G as a subgroup of the quaternions $\mathbf{H} = \mathbf{C} + \mathbf{C}j$, we get a 2 dimensional *complex* representation of G. Show that this is irreducible.
(b) Conclude the remaining irreducible representations of G are 1-dimensional. Find them. Write the character table for G.
(c) Let $G = D_8$ be the symmetries of the square. Show that G acts on $\mathbf{C}^2$ irreducibly, and determine the character table of G.
(d) Compare the character table of D_8 and Q_8. Comment??

2.4 Question

Determine the character table for D_{12}, the symmetry group of the hexagon.

2.5 Question

Determine the character table for D_{10}, the symmetry group of the 5-gon. For each representation of D_{10}, we may regard it as a represntation of $\mathbf{Z}/5 \leq D_{10}$. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of $\mathbf{Z}/5$.

2.6 Question

Construct the character tables for the group A_4.

2.7 Question

Describe the commutator subgroup of a group in terms of the character table of G.

2.8 Question

The table below is part of the character table of a finite group, but some of the rows are missing. (The columns are labelled by the order of the conjugacy class, $\gamma = \frac{1}{2}(-1 + i\sqrt{7})$, $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$.

	1	3	3	7	7
χ_a	1	1	1	ζ	$\overline{\zeta}$
χ_b	3	γ	$\overline{\gamma}$	0	0
χ_c	3	$\overline{\gamma}$	γ	0	0

(a) Complete the table.
(b) Describe the group in terms of generators and relations.

2.9 Question

(a) Let X and Y be finite sets acted on by G, and denote by $\mathbf{C}[X]$ and $\mathbf{C}[Y]$ the corresponding permutation representations. Show that $\dim \mathrm{Hom}_G(\mathbf{C}[X], \mathbf{C}[Y])$ is the number of G-orbits on $X \times Y$.
(b) Using part (a), find the multiplicity of the trivial representation in $\mathbf{C}[X]$.

2.10 Question

The symmetric group S_n acts on $\mathbf{C}^n$ by permuting the standard basis vectors. Show that it contains the a single copy of the trivial representation and that the complement V is irreducible.
The group S_n also acts on the set of 2-element subsets of $\{1, \ldots, n\}$. Call the associated permutation representation W. Show that, if $n > 3$, W contains a copy of the trivial rep, a copy of V, and that the remaining summand is irreducible.
Hint: Use Question 2.9 to compute $\|\chi_W\|^2$, $\langle 1|\chi_W\rangle$ *and* $\langle \chi_V|\chi_W\rangle$.

2.11 Question

Compute the character tables of the groups S_3, S_4, S_5.
Compute the character tables of the groups A_3, A_4, A_5.
The groups S_n act by conjugation on A_n. This induces an action on the set of irreducible representations of A_n. Describe it, for $n = 3, 4, 5$.

2.12 Question

Let p be a prime and $\mathbf{F}_p := \mathbf{Z}/p$. The group $\mathrm{SL}_2(\mathbf{F}_p)$ acts on the set $\mathbf{P}^1(\mathbf{F}_p) := \mathbf{F}_p \cup \{\infty\}$ by Möbius transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}.$$

Show that $\mathrm{SL}_2(\mathbf{F}_p)$ has an irreducible representation of dimension p. (This works for any finite field $\mathbf{F}_q$.)
Hint: Use question 2.9.

2.13 Question

Prove that the number of irreducible characters which take only real values is equal to the number of self-inverse conjugacy classes.
[A conjugacy class is self-inverse *if contains all inverses of its elements]*

2.14 Question

(a) Let U be an irreducible representation of G with character χ_U. Show that, for any irreducible representation $\rho : G \to \mathrm{GL}(W)$, the following linear operator is a scalar, and determine its value:

$$\sum_{g \in G} \chi_U(g^{-1})\rho(g) : W \to W.$$

[You should find that it is zero, unless W is isomorphic to U].
(b) Any representation V of G has a *canonical* decomposition $V = \bigoplus V_k$ into isotypical components. If χ is the character of V and χ_k the irreducible character associated to the summand V_k, show that the projection operator P_k of V onto the summand V_k is given by the formula

$$P_k = \frac{\chi_k(1)}{|G|} \sum_{g \in G} \chi_k(g^{-1})\rho(g).$$