Representation Theory, Sheet 2

CT, Lent 2005

G is a finite group and vector spaces are finite-dimensional over $\mathbb{C}$.

2.1 Question

Prove that there is a natural isomorphism (not requiring any choices) $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

2.2 Question

Let V be a finite-dimensional representation of G and let W_k, $k = 1, \ldots, r$ be a complete set of irreducible reps up to isomorphism. Show that we have a natural isomorphism of G-representations

$$V \cong \bigoplus_k \mathrm{Hom}^G(W_k, V) \otimes W_k.$$

In other words, $\mathrm{Hom}^G(W_k, V) \otimes W_k$ is naturally isomorphic to the W_k-isotypical component of V.

2.3 Question

Let ρ and σ be representations of two finite groups G and H on complex vector spaces V and W. Define a representation $\rho \otimes \sigma$ of the product group $G \times H$ on $V \otimes W$ by $(\rho \otimes \sigma)(g, h) := \rho(g) \otimes \sigma(h)$. Determine the character of $\rho \otimes \sigma$ and, using this, show that it is irreducible if ρ and σ are so.
How do you reconcile this with the example in class that the tensor square G-representation $W \otimes W$ can be *reducible*, even if W was irreducible?

2.4 Question

(a) Let X and Y be finite sets with G-action, and denote by $\mathbf{C}[X]$ and $\mathbf{C}[Y]$ the corresponding permutation representations. Show that $\dim \mathrm{Hom}^G(\mathbf{C}[X], \mathbf{C}[Y])$, the space of linear intertwiners, is the number of G-orbits on the Cartesian product $X \times Y$.
(b) Using part (a), find the multiplicity of the trivial representation in $\mathbf{C}[X]$.
(c) When the G acts transitively on X, formulate a criterion under which the complement in $\mathbf{C}[X]$ of the trivial representation is irreducible, in terms of the action on $X \times X$.

2.5 Question

Let p be a prime and $\mathbf{F}_p := \mathbf{Z}/p$. The group $\mathrm{SL}_2(\mathbf{F}_p)$ acts on the set $\mathbf{P}^1(\mathbf{F}_p) := \mathbf{F}_p \cup \{\infty\}$ by Möbius transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (z) = \frac{az + b}{cz + d}.$$

Show that $\mathrm{SL}_2(\mathbf{F}_p)$ has an irreducible representation of dimension p. (This works for any finite field $\mathbf{F}_q$.)
Hint: Use question 2.4.

2.6 Question

The symmetric group S_n acts on $\mathbf{C}^n$ by permuting the standard basis vectors. Show that it contains the a single copy of the trivial representation and that the complement V is irreducible.
The group S_n also acts on the set of 2-element subsets of $\{1, \ldots, n\}$. Call the associated permutation representation W. Show that, if $n > 3$, W contains a copy of the trivial rep, a copy of V, and that the remaining summand is irreducible.
Hint: Use Question 2.4 to compute $\|\chi_W\|^2$, $\langle 1|\chi_W\rangle$ and $\langle \chi_V|\chi_W\rangle$.

2.7 Question

(a) Let U be an irreducible representation of G with character χ_U. Show that, for any irreducible representation $\rho : G \to \mathrm{GL}(W)$, the following linear operator is a scalar, and determine its value:

$$\sum_{g \in G} \chi_U(g^{-1})\rho(g) : W \to W.$$

[You should find that it is zero, unless W is isomorphic to U].
(b) We saw that any representation V of G has a *canonical* decomposition $V = \bigoplus V_k$ into isotypical components. If χ is the character of V and χ_k the irreducible character associated to the summand V_k, show that the projection operator P_k of V onto the summand V_k is given by the formula

$$P_k = \frac{\chi_k(1)}{|G|} \sum_{g \in G} \chi_k(g^{-1})\rho(g).$$

2.8 Question

(a) Show that the complex character table of a finite group G is invertible when viewed as a matrix.
(b) Prove that the number of irreducible characters which take only real values is equal to the number of self-inverse conjugacy classes.
[A conjugacy class is self-inverse *if contains all inverses of its elements]*
(c) Conclude that an irreducible character takes only real values iff the corresponding representation admits an invariant bilinear form. Show, moreover, that this form is either symmetric or anti-symmetric.

2.9 Question

(a) Prove that the *Heisenberg group* H_n of 3×3 strictly upper-triangular matrices with entries in the ring $\mathbb{Z}/n$ is isomorphic to the group on three generators c, g, h with relations $c^n = g^n = h^n = 1$, $cg = gc$, $ch = hc$, $gh = chg$.
For the remaining parts of the question, you may assume that n is a prime.
(b) Determine the conjugacy classes.
(c) Let $\omega \neq 1$ be an nth root of unity. By computing the character or otherwise, show that the representation ρ of H_n on $\mathbf{C}^n$ in which $\rho(c) = \omega \cdot \mathrm{Id}$, $\rho(g) = \mathrm{diag}[1, \omega, \ldots, \omega^{n-1}]$, and $\rho(h)$ cycles through the standard basis vectors, is irreducible.
(d) Write the character table for H_n.
Hint: Find n^2 1-dimensional representations, and $(n-1)$ of dimension n.

2.10 Question

Prove the formulae for the characters of the symmetric and exterior squares of a representation, and derive formulae for the cubes.
Hint: For each $g \in G$, use a basis of V in which $\rho(g)$ is diagonal, and use our bases for S^kV and Λ^kV.

2.11 Question

Construct the character table for the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Also describe the unique higher-dimensional irreducible representation.

2.12 Question

Construct the character tables for the groups A_4 and S_4.

2.13 Question

Inspired by our calculation for A_5, construct the character table for S_5.