Representation Theory Sheet 1

CT, Lent 2005

G is a group; vector spaces are finite-dimensional and the field of scalars is $\mathbb{C}$, unless otherwise stated.

1.1 Question

Let ρ be a representation of G. Show that $\det \rho$ defines a 1-dimensional representation of G.

1.2 Question

Let $\theta : G \to \mathbb{C}^\times$ be a one-dimensional representation of G, and let ρ be another representation on a vector space V. Show that $\theta \otimes \rho : G \to GL(V)$, defined by $(\theta \otimes \rho)(g) := \theta(g) \cdot \rho(g)$, is also a representation, and is irreducible if ρ was so.

1.3 Question

Let $\rho : \mathbb{Z} \to GL(2;\mathbb{C})$ be the representation of $\mathbb{Z}$ defined by $\rho(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that ρ is not completely reducible.

1.4 Question

Let N be a normal subgroup of G. Relate the representations of G/N to the representations of G.

1.5 Question

Let G be a cyclic group of order n. Decompose the regular representation explicitly as a sum of one-dimensional representation, by giving the matrix of change of coordinates from the natural basis $\{\mathbf{e}_g\}_{g \in G}$ to one where the group action is diagonal.

1.6 Question

Let $\{M_\alpha\}$ be a collection of $N \times N$ matrices which are separately diagonalisable and commute pairwise. Show that they are simultaneously diagonalisable. (Requires patience.)

1.7 Question

The following question is quite easy, despite its length, but requires you to think through the definitions. Let G be a finite abelian group and denote by $\widehat{G}$ the set of isomorphism classes of irreducible G-representations.

(a) Show that $\widehat{G}$ forms an abelian group, under the operation of tensoring representations, as in Q. 2.

(b) If G is cyclic of order n, describe $\widehat{G}$.

(c) Fix $g \in G$. Show that sending a one-dimensional representation χ of G to the value $\chi(g) \in \mathbb{C}^\times$ defines a group homomorphism $\widehat{G} \to \mathbb{C}^\times$.

(d) Show that the assignment in (c) defines a homomorphism $G \to \widehat{\widehat{G}}$, and that this is in fact an isomorphism.

(e) Show that a homomorphism $\phi : G \to H$ of abelian groups induces a homomorphism $\widehat{\phi} : \widehat{H} \to \widehat{G}$, by sending $\chi : H \to \mathbb{C}^\times$ to $\widehat{\chi} := \chi \circ \phi$, and that $\widehat{\phi}$ is surjective iff ϕ is injective. Show that $\widehat{\widehat{\phi}} = \phi$.

Note: $\widehat{G}$ is called the dual group of G.

1.8 Question

(a) *(Weyl's unitary trick)* Let V be a representation of the finite group G and let $\langle \, | \, \rangle$ be any inner product on V. Show that the following averaged inner product $\langle \, | \, \rangle'$ is G-invariant:

$$\langle \mathbf{v}|\mathbf{w}\rangle' = \frac{1}{|G|} \sum\nolimits_{g\in G} \langle g\mathbf{v}|g\mathbf{w}\rangle.$$

(b) A *skew-symmetric bilinear form* is a bilinear map $V \times V \to \mathbb{C}$ satisfying $\langle \mathbf{v}|\mathbf{w}\rangle = -\langle \mathbf{w}|\mathbf{v}\rangle$, and is called non-degenerate if for any $\mathbf{v} \in V \setminus \{0\}$ there exists $\mathbf{w} \in V$ with $\langle \mathbf{v}|\mathbf{w}\rangle \neq 0$. Show that even-dimensional vector space carries a non-degenerate skew form. Can you conclude by arguing as in (a) that every even-dimensional representation of G carries an invariant such form? If not, find a counter-example.

1.9 Question

Let X be a finite set with G-action and ρ_X the associated "permutation representation" on the vector space $\mathbb{C}[X]$ of functions on X. Show that the value at $g \in G$ of the character of ρ_X is the number of fixed points of g in X.

1.10 Question

Let p be a prime number and G the cyclic group of order p, with generator g ($g^p = 1$). Show that the homomorphism $\rho : G \to GL(2; \mathbb{F}_p)$ defined by $\rho(g) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives a 2-dimensional representation of G over $\mathbb{F}_p$ which is *not* completely reducible, and hence the constraint on the characteristic in the complete reducibility theorem is necessary.

1.11 Question*

Recall that an *algebra* over a field k is a (not necessarily commutative) ring containing a distinguished copy of k which commutes with every ring element, and a *division algebra* is one where every non-zero element is invertible under multiplication. Prove that a finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

1.12 Question

Find examples of an irreducible representations of a group over the field of *real* numbers whose algebra of endomorphisms is (a) $\mathbb{R}$ (b) $\mathbb{C}$.

1.13 Question

Let the quaternion group $Q_8 := \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ act on the quaternions $\mathbb{H}$ by left multiplication.
(a) Show that this is an irreducible representation over $\mathbb{R}$ (with the natural $\mathbb{R}$-vector space structure on $\mathbb{H}$).
(b) Show that the endomorphism algebra of this representation is isomorphic to $\mathbb{H}$.

1.14 Question*

Classify the irreducible representations of the dihedral group D_{2n}.
Hint: Study first the cyclic subgroup $C_n \subset D_{2n}$, and try to see how the reflection must act. You should find that the parity of n matters.