## Probability and Measure 2

3.1. Suppose that a simple function $f$ has two representations

$$
f=\sum_{k=1}^{m} a_{k} 1_{A_{k}}=\sum_{j=1}^{n} b_{j} 1_{B_{j}} .
$$

For $\varepsilon=\left(\varepsilon_{1}, \ldots, \varepsilon_{m}\right) \in\{0,1\}^{m}$, define $A_{\varepsilon}=A_{1}^{\varepsilon_{1}} \cap \ldots \cap A_{m}^{\varepsilon_{m}}$ where $A_{k}^{0}=A_{k}^{c}$ and $A_{k}^{1}=A_{k}$. Define similarly $B_{\delta}$ for $\delta \in\{0,1\}^{n}$. Then set $f_{\varepsilon, \delta}=\sum_{k=1}^{m} \varepsilon_{k} a_{k}$ if $A_{\varepsilon} \cap B_{\delta} \neq \emptyset$ and $f_{\varepsilon, \delta}=0$ otherwise. Show that, for any measure $\mu$,

$$
\sum_{k=1}^{m} a_{k} \mu\left(A_{k}\right)=\sum_{\varepsilon, \delta} f_{\varepsilon, \delta} \mu\left(A_{\varepsilon} \cap B_{\delta}\right)
$$

and deduce that

$$
\sum_{k=1}^{m} a_{k} \mu\left(A_{k}\right)=\sum_{j=1}^{n} b_{j} \mu\left(B_{j}\right) .
$$

3.2. Let $\mu$ and $\nu$ be finite Borel measures on $\mathbb{R}$. Let $f$ be a continuous bounded function on $\mathbb{R}$. Show that $f$ is integrable with respect to $\mu$ and $\nu$. Show further that, if $\mu(f)=\nu(f)$ for all such $f$, then $\mu=\nu$.
3.3. Let $f$ be an integrable function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that, for some $\pi$-system $\mathcal{A}$ containing $E$ and generating $\mathcal{E}$, we have $\mu\left(f 1_{A}\right)=0$ for all $A \in \mathcal{A}$. Show that $f=0$ a.e.
3.4. Let $X$ be a non-negative integer-valued random variable. Show that

$$
\mathbb{E}(X)=\sum_{n=1}^{\infty} \mathbb{P}(X \geq n)
$$

Deduce that, if $\mathbb{E}(X)=\infty$ and $X_{1}, X_{2}, \ldots$ is a sequence of independent random variables with the same distribution as $X$, then, almost surely, $\lim \sup _{n}\left(X_{n} / n\right) \geq 1$, and moreover $\lim \sup _{n}\left(X_{n} / n\right)=$ $\infty$.

Now suppose that $Y_{1}, Y_{2}, \ldots$ is any sequence of independent identically distributed random variables with $\mathbb{E}\left|Y_{1}\right|=\infty$. Show that, almost surely, $\lim \sup _{n}\left(\left|Y_{n}\right| / n\right)=\infty$, and moreover $\lim \sup _{n}\left(\left|Y_{1}+\cdots+Y_{n}\right| / n\right)=\infty$.
3.5. For $\alpha \in(0, \infty)$ and $x \in(0, \infty)$, define $f_{\alpha}(x)=x^{-\alpha}$. Show that $f_{\alpha}$ is integrable with respect to Lebesgue measure on $(0,1]$ if and only if $\alpha<1$. Show also that $f_{\alpha}$ is integrable with respect to Lebesgue measure on $[1, \infty)$ if and only if $\alpha>1$.
3.6. Show that the function $\sin x / x$ is not Lebesgue integrable over $[1, \infty)$ but that integral $\int_{1}^{N}(\sin x / x) d x$ converges as $N \rightarrow \infty$.
3.7. Show that, as $n \rightarrow \infty$,

$$
\int_{0}^{\infty} \sin \left(e^{x}\right) /\left(1+n x^{2}\right) d x \rightarrow 0 \quad \text { and } \quad \int_{0}^{1}(n \cos x) /\left(1+n^{2} x^{\frac{3}{2}}\right) d x \rightarrow 0
$$

3.8. Let $u$ and $v$ be differentiable functions on $\mathbb{R}$ with continuous derivatives $u^{\prime}$ and $v^{\prime}$. Suppose that $u v^{\prime}$ and $u^{\prime} v$ are integrable on $\mathbb{R}$ and $u(x) v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Show that

$$
\int_{\mathbb{R}} u(x) v^{\prime}(x) d x=-\int_{\mathbb{R}} u^{\prime}(x) v(x) d x .
$$

3.9. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces and let $f: E \rightarrow G$ be a measurable function. Given a measure $\mu$ on $(E, \mathcal{E})$, consider the image measure $\nu=\mu \circ f^{-1}$ on $(G, \mathcal{G})$. Show that $\nu(g)=\mu(g \circ f)$ for all non-negative measurable functions $g$ on $G$.
3.10. The moment generating function $\phi$ of a real-valued random variable $X$ is defined by $\phi(\theta)=$ $\mathbb{E}\left(e^{\theta X}\right), \quad \theta \in \mathbb{R}$.

Suppose that $\phi$ is finite on an open interval containing 0 . Show that $\phi$ has derivatives of all orders at 0 and that $X$ has finite moments of all orders given by

$$
\mathbb{E}\left(X^{n}\right)=\left.\left(\frac{d}{d \theta}\right)^{n}\right|_{\theta=0} \phi(\theta) .
$$

3.11. Let $X_{1}, \ldots, X_{n}$ be random variables with density functions $f_{1}, \ldots, f_{n}$ respectively. Suppose that the $\mathbb{R}^{n}$-valued random variable $X=\left(X_{1}, \ldots, X_{n}\right)$ also has a density function $f$. Show that $X_{1}, \ldots, X_{n}$ are independent if and only if

$$
f\left(x_{1}, \ldots, x_{n}\right)=f_{1}\left(x_{1}\right) \ldots f_{n}\left(x_{n}\right) \quad \text { a.e. }
$$

3.12. Show that, for all non-negative measurable functions $f$ on $[0, \infty)$, the function $(x, y) \mapsto$ $f(|(x, y)|)$ is measurable on $\mathbb{R}^{2}$ and (without using the Jacobian formula)

$$
\int_{\mathbb{R}^{2}} f(|(x, y)|) d x d y=2 \pi \int_{0}^{\infty} r f(r) d r .
$$

Hence show that $(2 \pi)^{-1 / 2} e^{-x^{2} / 2}$ is a probability density function.
3.13. Let $\mu$ and $\nu$ be probability measures on $(E, \mathcal{E})$ and let $f: E \rightarrow[0, R]$ be a measurable function. Suppose that $\nu(A)=\mu\left(f 1_{A}\right)$ for all $A \in \mathcal{E}$. Let $\left(X_{n}: n \in \mathbb{N}\right)$ be a sequence of independent random variables in $E$ with law $\mu$ and let $\left(U_{n}: n \in \mathbb{N}\right)$ be a sequence of independent $U[0,1]$ random variables. Set

$$
T=\min \left\{n \in \mathbb{N}: R U_{n} \leq f\left(X_{n}\right)\right\}, \quad Y=X_{T} .
$$

Show that $Y$ has law $\nu$. (This justifies simulation by rejection sampling.)

