

### Probability and Measure 4

**7.8.** Let  $(X_n : n \in \mathbb{N})$  be a sequence of random variables in  $\mathbb{R}$  and let  $X$  be another such random variable. Show that  $X_n \rightarrow X$  weakly if and only if  $X_n \rightarrow X$  in distribution.

**7.9.** Let  $\mu$  be a Borel probability measure on  $\mathbb{R}^d$  and let  $(\mu_n : n \in \mathbb{N})$  be a sequence of such measures. Suppose that  $\mu_n(f) \rightarrow \mu(f)$  for all  $C^\infty$  functions on  $\mathbb{R}^d$  of compact support. Show that  $\mu_n$  converges weakly to  $\mu$  on  $\mathbb{R}^d$ .

**8.1.** Let  $X = (X_1, \dots, X_n)$  be a Gaussian random variable in  $\mathbb{R}^n$  with mean  $\mu$  and covariance matrix  $V$ . Assume that  $V$  is invertible write  $V^{-1/2}$  for the positive-definite square root of  $V^{-1}$ . Set  $Y = (Y_1, \dots, Y_n) = V^{-1/2}(X - \mu)$ . Show that  $Y_1, \dots, Y_n$  are independent  $N(0, 1)$  random variables. Show further that we can write  $X_2$  in the form  $X_2 = aX_1 + Z$  where  $Z$  is independent of  $X_1$  and determine the distribution of  $Z$ .

**8.2.** Let  $X_1, \dots, X_n$  be independent  $N(0, 1)$  random variables. Show that

$$\left( \bar{X}, \sum_{m=1}^n (X_m - \bar{X})^2 \right) \quad \text{and} \quad \left( \frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2 \right)$$

have the same distribution, where  $\bar{X} = (X_1 + \dots + X_n)/n$ .

**9.1.** Let  $(E, \mathcal{E}, \mu)$  be a measure space and  $\tau : E \rightarrow E$  a measure-preserving transformation. Show that  $\mathcal{E}_\tau := \{A \in \mathcal{E} : \tau^{-1}(A) = A\}$  is a  $\sigma$ -algebra, and that a measurable function  $f$  is  $\mathcal{E}_\tau$ -measurable if and only if it is *invariant*, that is  $f \circ \tau = f$ .

**9.2.** Show that, if  $\theta$  is an ergodic measure-preserving transformation and  $f$  is a  $\theta$ -invariant function, then there exists a constant  $c \in \mathbb{R}$  such that  $f = c$  a.e..

**9.3.** For  $x \in [0, 1)$ , set  $\tau(x) = 2x \bmod 1$ . Show that  $\tau$  is a measure-preserving transformation of  $([0, 1), \mathcal{B}([0, 1)), dx)$ , and that  $\tau$  is ergodic. Identify the invariant function  $\bar{f}$  corresponding to each integrable function  $f$ .

**9.4.** Fix  $a \in [0, 1)$  and define, for  $x \in [0, 1)$ ,  $\tau(x) = x + a \bmod 1$ . Show that  $\tau$  is also a measure-preserving transformation of  $([0, 1), \mathcal{B}([0, 1)), dx)$ . Determine for which values of  $a$  the transformation  $\tau$  is ergodic. *Hint: you may use the fact that any integrable function  $f$  on  $[0, 1)$  whose Fourier coefficients all vanish must itself vanish a.e..* Identify, for all values of  $a$ , the invariant function  $\bar{f}$  corresponding to an integrable function  $f$ .

**9.5.** Call a sequence of random variables  $(X_n : n \in \mathbb{N})$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  *stationary* if for each  $n, k \in \mathbb{N}$  the random vectors  $(X_1, \dots, X_n)$  and  $(X_{k+1}, \dots, X_{k+n})$  have the same distribution: for  $A_1, \dots, A_n \in \mathcal{B}$ ,

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \dots, X_{k+n} \in A_n).$$

Show that, if  $(X_n : n \in \mathbb{N})$  is a stationary sequence and  $X_1 \in L^p$ , for some  $p \in [1, \infty)$ , then

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow X \quad \text{a.s. and in } L^p,$$

for some random variable  $X \in L^p$  and find  $\mathbb{E}(X)$ .

**10.1.** Let  $(X_n : n \in \mathbb{N})$  be a sequence of independent random variables, such that  $\mathbb{E}(X_n) = \mu$  and  $\mathbb{E}(X_n^4) \leq M$  for all  $n$ , for some constants  $\mu \in \mathbb{R}$  and  $M < \infty$ . Set  $P_n = X_1 X_2 + X_2 X_3 + \dots + X_{n-1} X_n$ . Show that  $P_n/n$  converges a.s. as  $n \rightarrow \infty$  and identify the limit.

**10.2.** The Cauchy distribution has density function  $f(x) = \pi^{-1}(1+x^2)^{-1}$  for  $x \in \mathbb{R}$ . Show that the corresponding characteristic function is given by  $\varphi(u) = e^{-|u|}$ . Show also that, if  $X_1, \dots, X_n$  are independent Cauchy random variables, then the random variable  $(X_1 + \dots + X_n)/n$  is also Cauchy.

**10.3.** Let  $f$  be a bounded continuous function on  $(0, \infty)$ , having Laplace transform

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx, \quad \lambda \in (0, \infty).$$

Let  $(X_n : n \in \mathbb{N})$  be a sequence of independent exponential random variables, of parameter  $\lambda$ . Show that  $\hat{f}$  has derivatives of all orders on  $(0, \infty)$  and that, for all  $n \in \mathbb{N}$ , for some  $C(\lambda, n) \neq 0$  independent of  $f$ , we have

$$(d/d\lambda)^{n-1} \hat{f}(\lambda) = C(\lambda, n) \mathbb{E}(f(S_n))$$

where  $S_n = X_1 + \dots + X_n$ . Deduce that if  $\hat{f} \equiv 0$  then also  $f \equiv 0$ .

**10.4.** For each  $n \in \mathbb{N}$ , there is a unique probability measure  $\mu_n$  on the unit sphere  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$  such that  $\mu_n(A) = \mu_n(UA)$  for all Borel sets  $A$  and all orthogonal  $n \times n$  matrices  $U$ . Fix  $k \in \mathbb{N}$  and, for  $n \geq k$ , let  $\gamma_n$  denote the probability measure on  $\mathbb{R}^k$  which is the law of  $\sqrt{n}(x^1, \dots, x^k)$  under  $\mu_n$ . Show

(a) if  $X \sim N(0, I_n)$  then  $X/|X| \sim \mu_n$ ,

(b) if  $(X_n : n \in \mathbb{N})$  is a sequence of independent  $N(0, 1)$  random variables and if  $R_n = \sqrt{X_1^2 + \dots + X_n^2}$  then  $R_n/\sqrt{n} \rightarrow 1$  a.s.,

(c)  $\gamma_n$  converges weakly to the standard Gaussian distribution on  $\mathbb{R}^k$  as  $n \rightarrow \infty$ .