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Probability and Measure 1

1.1. Let *E* be a set and let *S* be a set of σ -algebras on *E*. Define

 $\mathcal{E}^* = \{ A \subseteq E : A \in \mathcal{E} \quad \text{for all} \quad \mathcal{E} \in \mathcal{S} \}.$

Show that \mathcal{E}^* is a σ -algebra on E. Show, on the other hand, by example, that the union of two σ -algebras on the same set need not be a σ -algebra.

1.2. Show that the following sets of subsets of \mathbb{R} all generate the same σ -algebra: (a) $\{(a,b): a < b\}$, (b) $\{(a,b]: a < b\}$, (c) $\{(-\infty,b]: b \in \mathbb{R}\}$.

1.3. Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4. Show that a π -system which is also a *d*-system is a σ -algebra.

1.5. Let μ be a finite-valued additive set function on a ring \mathcal{A} . Show that μ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in \mathcal{A} , with $\bigcap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6. Let (E, \mathcal{E}, μ) be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in \mathcal{E} , $\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events A_n are independent if and only if the σ -algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8. Let *B* be a Borel subset of the interval [0, 1]. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \triangle B$ $(= (A^c \cap B) \cup (A \cap B^c))$ is less than ε . Show further that this remains true for every Borel set in \mathbb{R} of finite Lebesgue measure.

1.9. Let (E, \mathcal{E}, μ) be a measure space. Call a subset $N \subseteq E$ null if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write \mathcal{N} for the set of null sets. Prove that the set of subsets $\mathcal{E}^{\mu} = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ -algebra and show that μ has a well-defined and countably additive extension to \mathcal{E}^{μ} given by $\mu(A \cup N) = \mu(A)$. We call \mathcal{E}^{μ} the completion of \mathcal{E} with respect to μ . Suppose now that E is σ -finite and write μ^* for the outer measure associated to μ , as in the proof of Carathéodory's Extension Theorem. Show that \mathcal{E}^{μ} is exactly the set of μ^* -measurable sets.

2.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on a measurable space (E, \mathcal{E}) . Show that the following functions are also measurable: $f_1 + f_2$, $f_1 f_2$, $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$, $\limsup_n f_n$, $\lim_n f_n$, \lim_n

2.2. Let (E, \mathcal{E}) and (G, \mathcal{G}) be measurable spaces, let μ be a measure on \mathcal{E} , and let $f : E \to G$ be a measurable function. Show that we can define a measure ν on \mathcal{G} by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$.

2.3. Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$ for all $x, y \in \mathbb{R}$.

2.4. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 \mathbb{1}_{A_n} - \mathbb{1}$ and set $\overline{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\overline{X}_n) = 0$ for all n, but that $\overline{X}_n \to -1$ almost surely as $n \to \infty$.

2.5. The zeta function is defined for s > 1 by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent random variables with

$$\mathbb{P}(X=n) = \mathbb{P}(Y=n) = n^{-s}/\zeta(s)$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_{p} \left(1 - \frac{1}{p^s} \right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write *H* for the highest common factor of *X* and *Y*. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

2.6. Let $(X_n : n \in \mathbb{N})$ be independent N(0, 1) random variables. Prove that

$$\limsup_{n} \left(X_n / \sqrt{2 \log n} \right) = 1 \quad \text{a.s.}$$

2.7. Let C_n denote the *n*th approximation to the Cantor set C: thus $C_0 = [0, 1], C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n . Show that

- (a) C is uncountable and has Lebesgue measure 0,
- (b) for all $x \in [0, 1]$, the limit $F(x) = \lim_{n \to \infty} F_n(x)$ exists,
- (c) the function F is continuous on [0, 1], with F(0) = 0 and F(1) = 1,
- (d) for almost all $x \in [0, 1]$, F is differentiable at x with F'(x) = 0.

Hint: express F_{n+1} *recursively in terms of* F_n *and use this relation to obtain a uniform estimate on* $F_{n+1} - F_n$.