Probability and Measure 3

4.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of integrable functions and suppose that $f_n \to f$ a.e. for some integrable function f. Show that, if $\|f_n\|_1 \to \|f\|_1$, then $\|f_n - f\|_1 \to 0$.

4.2. Let X be a random variable and let $1 \leq p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-p})$ as $\lambda \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-q})$ as $\lambda \to \infty$, then $X \in L^p(\mathbb{P})$.

4.3. Give a simple proof of Schwarz' inequality $\|fg\|_1 \leq \|f\|_2\|g\|_2$ for measurable functions f and g.

4.4. Show that $\|XY\|_1 = \|X\|_1\|Y\|_1$ for independent random variables X and Y. Show further that, if X and Y are also integrable, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

4.5. A *stepfunction* $f : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$: that is, for all $f \in L^p(\mathbb{R})$ and all $\varepsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \varepsilon$. Deduce that the set of continuous functions of compact support is also dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

4.6. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0 \quad$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2}\max_{k\leq n}|X_k| \to 0$ in probability.

5.1. Let $(E, \mathcal{E}, \mu)$ be a measure space and let $V_1 \leq V_2 \leq \dots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E, \mathcal{E}, \mu)$ for $f \in L^2$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in L^2.

5.2. Let $X = (X_1, \dots, X_n)$ be a random variable, with all components in $L^2(\mathbb{P})$. The covariance matrix $\mathrm{var}(X) = (c_{ij} : 1 \leq i, j \leq n)$ of X is defined by $c_{ij} = \mathrm{cov}(X_i, X_j)$. Show that $\mathrm{var}(X)$ is a non-negative definite matrix.

6.1. Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

6.2. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that

$$\mathbb{E}\big(\max_{k\leq n}|X_k|\big)/\sqrt{n} \to 0 \qquad \text{as} \quad n \to \infty.$$

7.1. Let $u, v \in L^1(\mathbb{R}^d)$ and define $f : \mathbb{R}^d \to \mathbb{C}$ by $f(x) = u(x) + iv(x)$. Set

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} u(x)dx + i \int_{\mathbb{R}^d} v(x)dx.$$

Show that, for all $y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} f(x-y)dx = \int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(-x)dx$$

and show that

$$\left| \int_{\mathbb{R}^d} f(x)dx \right| \leq \int_{\mathbb{R}^d} |f(x)|dx.$$

7.2. Show that the Fourier transform of a finite Borel measure on $\mathbb{R}^d$ is a bounded continuous function.

7.3. Determine which of the following distributions on $\mathbb{R}$ have an integrable characteristic function: $\mathrm{N}(\mu, \sigma^2)$, $\mathrm{Bin}(N, p)$, $\mathrm{Poisson}(\lambda)$, $\mathrm{U}[0,1]$.

7.4. For a finite Borel measure μ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a kth continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).$$

7.5. Define a function ψ on $\mathbb{R}$ by setting $\psi(x) = C \exp\{-(1-x^2)^{-1}\}$ for $|x| < 1$ and $\psi(x) = 0$ otherwise, where C is a constant chosen so that $\int_{\mathbb{R}} \psi(x)dx = 1$. For $f \in L^1(\mathbb{R})$ of compact support, show that $f * \psi$ is C^∞ and of compact support.

7.6. (i) Show that for any real numbers a, b one has $\int_a^b e^{itx} dx \to 0$ as $|t| \to \infty$.

(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$$

tends to 0 as $|t| \to \infty$. This is the *Riemann–Lebesgue Lemma.*

7.7. Say that $f \in L^2(\mathbb{R})$ is L^2-differentiable with L^2-derivative Df if

$$\|\tau_h f - f - hDf\|_2 / h \to 0 \quad \text{as} \quad h \to 0,$$

where $\tau_h f(x) = f(x+h)$. Show that the function $f(x) = \max(1-|x|, 0)$ is L^2-differentiable and find its L^2-derivative.

Suppose that $f \in L^1 \cap L^2$ is L^2-differentiable. Show that $u\hat{f}(u) \in L^2$. Deduce that f has a continuous version and that $\|f\|_\infty \leq C\|(1+|u|)\hat{f}(u)\|_2$ for some absolute constant $C < \infty$, to be determined. This is a simple example of a *Sobolev inequality.*