## Probability and Measure 2

**3.1.** Suppose that a simple function f has two representations

$$f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{i=1}^{n} b_k 1_{B_k}.$$

For  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$ , define  $A_{\varepsilon} = A_1^{\varepsilon_1} \cap \dots \cap A_m^{\varepsilon_m}$  where  $A_k^0 = A_k^c$  and  $A_k^1 = A_k$ . Define similarly  $B_{\delta}$  for  $\delta \in \{0, 1\}^n$ . Then set  $f_{\varepsilon, \delta} = \sum_{k=1}^m \varepsilon_k a_k$  if  $A_{\varepsilon} \cap B_{\delta} \neq \emptyset$  and  $f_{\varepsilon, \delta} = 0$  otherwise. Show that, for any measure  $\mu$ ,

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon, \delta} f_{\varepsilon, \delta} \mu(A_{\varepsilon} \cap B_{\delta})$$

and deduce that

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j).$$

**3.2.** Let  $\mu$  and  $\nu$  be finite Borel measures on  $\mathbb{R}$ . Let f be a continuous bounded function on  $\mathbb{R}$ . Show that f is integrable with respect to  $\mu$  and  $\nu$ . Show further that, if  $\mu(f) = \nu(f)$  for all such f, then  $\mu = \nu$ .

**3.3.** Let f be an integrable function on a measure space  $(E, \mathcal{E}, \mu)$ . Suppose that, for some  $\pi$ -system  $\mathcal{A}$  containing E and generating  $\mathcal{E}$ , we have  $\mu(f1_A) = 0$  for all  $A \in \mathcal{A}$ . Show that f = 0 a.e.

**3.4.** Let X be a non-negative integer-valued random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \ge n).$$

Deduce that, if  $\mathbb{E}(X) = \infty$  and  $X_1, X_2, \ldots$  is a sequence of independent random variables with the same distribution as X, then, almost surely,  $\limsup_n (X_n/n) \ge 1$ , and moreover  $\limsup_n (X_n/n) = \infty$ .

Now suppose that  $Y_1, Y_2, ...$  is any sequence of independent identically distributed random variables with  $\mathbb{E}|Y_1| = \infty$ . Show that, almost surely,  $\limsup_n (|Y_n|/n) = \infty$ , and moreover  $\limsup_n (|Y_1| + \cdots + |Y_n|/n) = \infty$ .

**3.5.** For  $\alpha \in (0, \infty)$  and  $x \in (0, \infty)$ , define  $f_{\alpha}(x) = x^{-\alpha}$ . Show that  $f_{\alpha}$  is integrable with respect to Lebesgue measure on (0, 1] if and only if  $\alpha < 1$ . Show also that  $f_{\alpha}$  is integrable with respect to Lebesgue measure on  $[1, \infty)$  if and only if  $\alpha > 1$ .

1

- **3.6.** Show that the function  $\sin x/x$  is not Lebesgue integrable over  $[1, \infty)$  but that integral  $\int_1^N (\sin x/x) dx$  converges as  $N \to \infty$ .
- **3.7.** Show that, as  $n \to \infty$ ,

$$\int_0^\infty \sin(e^x)/(1+nx^2)dx \to 0 \quad \text{ and } \quad \int_0^1 (n\cos x)/(1+n^2x^{\frac{3}{2}})dx \to 0.$$

**3.8.** Let u and v be differentiable functions on  $\mathbb{R}$  with continuous derivatives u' and v'. Suppose that uv' and u'v are integrable on  $\mathbb{R}$  and  $u(x)v(x) \to 0$  as  $|x| \to \infty$ . Show that

$$\int_{\mathbb{R}} u(x)v'(x)dx = -\int_{\mathbb{R}} u'(x)v(x)dx.$$

- **3.9.** Let  $(E,\mathcal{E})$  and  $(G,\mathcal{G})$  be measurable spaces and let  $f:E\to G$  be a measurable function. Given a measure  $\mu$  on  $(E,\mathcal{E})$ , consider the image measure  $\nu=\mu\circ f^{-1}$  on  $(G,\mathcal{G})$ . Show that  $\nu(g)=\mu(g\circ f)$  for all non-negative measurable functions g on G.
- **3.10.** The moment generating function  $\phi$  of a real-valued random variable X is defined by  $\phi(\theta) = \mathbb{E}(e^{\theta X}), \quad \theta \in \mathbb{R}.$

Suppose that  $\phi$  is finite on an open interval containing 0. Show that  $\phi$  has derivatives of all orders at 0 and that X has finite moments of all orders given by

$$\mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n \bigg|_{\theta=0} \phi(\theta).$$

**3.11.** Let  $X_1, \ldots, X_n$  be random variables with density functions  $f_1, \ldots, f_n$  respectively. Suppose that the  $\mathbb{R}^n$ -valued random variable  $X = (X_1, \ldots, X_n)$  also has a density function f. Show that  $X_1, \ldots, X_n$  are independent if and only if

$$f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$$
 a.e.

**3.12.** Show that, for all non-negative measurable functions f on  $[0, \infty)$ , the function  $(x, y) \mapsto f(|(x, y)|)$  is measurable on  $\mathbb{R}^2$  and (without using the Jacobian formula)

$$\int_{\mathbb{R}^2} f(|(x,y)|) dx dy = 2\pi \int_0^\infty r f(r) dr.$$

Hence show that  $(2\pi)^{-1/2}e^{-x^2/2}$  is a probability density function.

**3.13.** Let  $\mu$  and  $\nu$  be probability measures on  $(E, \mathcal{E})$  and let  $f : E \to [0, R]$  be a measurable function. Suppose that  $\nu(A) = \mu(f1_A)$  for all  $A \in \mathcal{E}$ . Let  $(X_n : n \in \mathbb{N})$  be a sequence of independent random variables in E with law  $\mu$  and let  $(U_n : n \in \mathbb{N})$  be a sequence of independent U[0, 1] random variables. Set

$$T = \min\{n \in \mathbb{N} : RU_n \le f(X_n)\}, \qquad Y = X_T.$$

Show that Y has law  $\nu$ . (This justifies simulation by rejection sampling.)